

PRINCIPALIZATION OF 2-CLASS GROUPS OF TYPE $(2, 2, 2)$ OF BIQUADRATIC FIELDS $\mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-1})$

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ABSTRACT. Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{q}\right) = \left(\frac{p_2}{q}\right) = -1$. Put $d = p_1 p_2 q$ and $i = \sqrt{-1}$, then the bicyclic biquadratic field $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ has an elementary abelian 2-class group, $\mathbf{Cl}_2(\mathbb{k})$, of rank 3. In this paper, we study the principalization of the 2-classes of \mathbb{k} in its fourteen unramified abelian extensions \mathbb{K}_j and \mathbb{L}_j within $\mathbb{k}_2^{(1)}$, that is the Hilbert 2-class field of \mathbb{k} . We determine the nilpotency class, the coclass, generators and the structure of the metabelian Galois group $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of the second Hilbert 2-class field $\mathbb{k}_2^{(2)}$ of \mathbb{k} . Additionally, the abelian type invariants of the groups $\mathbf{Cl}_2(\mathbb{K}_j)$ and $\mathbf{Cl}_2(\mathbb{L}_j)$ and the length of the 2-class tower of \mathbb{k} are given.

1. INTRODUCTION

The 2-class tower of imaginary quadratic fields $k = \mathbb{Q}(\sqrt{d})$ with 2-class group of type $(2, 2, 2)$ has been investigated by E. Benjamin, F. Lemmermeyer, and C. Snyder [10]. More recently, H. Nover [23] provided evidence of such towers with exactly three stages. However, nothing was known about base fields of higher degree until A. Azizi, A. Zekhnini, and M. Taous focussed on complex bicyclic biquadratic fields $\mathbb{k} = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$, which are called *special Dirichlet fields* by D. Hilbert [13]. In [8], they determined the maximal unramified pro-2 extension of special Dirichlet fields with 2-class group of type $(2, 2, 2)$ and radicand $d = 2p_1 p_2$, where $p_1 \equiv p_2 \equiv 5 \pmod{8}$ denote primes. It is the purpose of the present article to pursue this research project further for special Dirichlet fields with $\mathbf{Cl}_2(\mathbb{k})$ of type $(2, 2, 2)$ and radicand $d = p_1 p_2 q$ composed of primes $p_1 \equiv p_2 \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$. As predicted in [20, § 4.2, pp. 451–452], the particular

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feature of the lattice of intermediate fields between a base field \mathbb{k} with 2-class group of type $(2, 2, 2)$ and its Hilbert 2-class field $\mathbb{k}_2^{(1)}$ is the constitution by *two layers* of unramified abelian extensions, rather than by just one layer as for a 2-class group of type $(2, 2)$, and it seems to be the first time that complete results are given here for the second layer. The layout of this paper is the following. First, all main theorems are presented in § 2. To be able to compute the Hasse unit index $Q_K = [E_K : W_K E_{K^+}]$ of CM-fields K with maximal real subfield K^+ and the unit index $q(K/\mathbb{Q}) = [E_K : \prod_{j=1}^s E_{k_j}]$ of multiquadratic fields K with quadratic subfields k_j , preliminary results on fundamental systems of units (FSU) of such fields are summarized in § 3. In § 4, one of the seven unramified quadratic extensions of \mathbb{k} , denoted by $\mathbb{K}_3 = \mathbb{k}(\sqrt{q})$, turns out to carry crucial information about the Galois group $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of the second Hilbert 2-class field $\mathbb{k}_2^{(2)}$ of \mathbb{k} , since $|G| = 2 \cdot h(\mathbb{K}_3) \geq 64$. The fact that \mathbb{K}_3 , which is contained in the genus field $\mathbb{k}^{(*)}$ of \mathbb{k} , has an abelian 2-class tower permits the conclusion that \mathbb{k} has a metabelian 2-class tower of exact length 2. The abelian type invariants of $\mathbf{Cl}_2(\mathbb{K}_j)$, in dependence on the 2-class numbers of the subfields $\mathbb{Q}(\sqrt{p_1 p_2})$ and $\mathbb{Q}(\sqrt{-p_1 p_2})$ of \mathbb{K}_3 and on the Legendre symbol $\left(\frac{p_1}{p_2}\right)$, are determined in § 5. Explicit generators, in the form of Artin symbols, a presentation of the second 2-class group G , and the structure of $G' \simeq \mathbf{Cl}_2(\mathbb{k}_2^{(1)})$, in dependence on the norm of the fundamental unit of $\mathbb{Q}(\sqrt{p_1 p_2})$, are also given in § 5. The nilpotency class $c(G)$ and the coclass $cc(G)$ turn out to depend on biquadratic residue symbols for p_1 and p_2 . The principalization kernels $\kappa_{\mathbb{K}_j/\mathbb{k}}$ can be given either in terms of generators of $\mathbf{Cl}_2(\mathbb{k})$ or as norm class groups. They are determined as kernels of Artin transfers $V_{G, G_j} : G/G' \rightarrow G_j/G'_j$ from G to its maximal subgroups G_j , $1 \leq j \leq 7$, using the presentation of G . These invariants form the *transfer kernel type* (TKT) of G in [20, Dfn. 1.1, p. 403]. Finally, the abelian type invariants of $\mathbf{Cl}_2(\mathbb{L}_j)$, $1 \leq j \leq 7$, are calculated as abelian quotient invariants $\mathcal{G}_j/\mathcal{G}'_j$ of the subgroups \mathcal{G}_j of index $[G : \mathcal{G}_j] = 4$ of G , which make up the *transfer target type* (TTT) of G in [20, Dfn. 1.1]. We conclude with numerical examples, statistics, and details about the groups G in § 6.

Let m be a square-free integer and K be a number field. Throughout this paper, we adopt the following notation:

- $h(m)$, resp. $h(K)$: the 2-class number of $\mathbb{Q}(\sqrt{m})$, resp. K .
- ε_m : the fundamental unit of $\mathbb{Q}(\sqrt{m})$, if $m > 0$.
- \mathcal{O}_K : the ring of integers of K .
- E_K : the unit group of \mathcal{O}_K .
- W_K : the group of roots of unity contained in K .
- ω_K : the order of W_K .
- $i = \sqrt{-1}$.
- K^+ : the maximal real subfield of K , if K is a CM-field.
- $Q_K = [E_K : W_K E_{K^+}]$ is the Hasse unit index, if K is a CM-field.

- $q(K/\mathbb{Q}) = [E_K : \prod_j^s E_{k_j}]$ is the unit index of K , if K is multiquadratic and k_j are the quadratic subfields of K .
- $K^{(*)}$: the absolute genus field of K (over \mathbb{Q}).
- $\mathbf{Cl}_2(K)$: the 2-class group of K .
- $\kappa_{L/K}$: the subgroup of classes of $\mathbf{Cl}_2(K)$ which become principal in an extension L/K (the *principalization-* or *capitulation-kernel* of L/K).

2. MAIN RESULTS

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the following conditions (1):

$$\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{q}\right) = \left(\frac{p_2}{q}\right) = -1. \quad (1)$$

Then there exist some positive integers e, f, g and h such that $p_1 = e^2 + 4f^2$ and $p_2 = g^2 + 4h^2$. Put $p_1 = \pi_1\pi_2$ and $p_2 = \pi_3\pi_4$, where $\pi_1 = e + 2if$ and $\pi_2 = e - 2if$ (resp. $\pi_3 = g + 2ih$ and $\pi_4 = g - 2ih$) are conjugate prime elements in the cyclotomic field $k = \mathbb{Q}(i)$ dividing p_1 (resp. p_2). Denote by \mathbb{k} the complex bicyclic biquadratic field $\mathbb{Q}(\sqrt{d}, i)$, where $d = p_1p_2q$. Its three quadratic subfields are $k = \mathbb{Q}(i)$, $k_0 = \mathbb{Q}(\sqrt{d})$ and $\bar{k}_0 = \mathbb{Q}(\sqrt{-d})$. Let $\mathbb{k}_2^{(1)}$ be the Hilbert 2-class field of \mathbb{k} , $\mathbb{k}_2^{(2)}$ be its second Hilbert 2-class field and G be the Galois group of $\mathbb{k}_2^{(2)}/\mathbb{k}$. According to [5], \mathbb{k} has an elementary abelian 2-class group $\mathbf{Cl}_2(\mathbb{k})$ of rank 3, that is, of type (2, 2, 2). The main results of this paper are Theorems 2, 3 and 4 below; whereas Theorem 1 is proved in [6], using [5] and [14].

2.1. Unramified extensions of \mathbb{k} . The first and the second assertion of the following theorem hold according to [14] and [5] respectively, the others are proved in [6]. The fields in Theorem 1 are visualized in Figure 1.

Theorem 1. *Let p_1, p_2 and q be different primes as specified by equation (1).*

- (1) *The 2-class groups of k_0, \bar{k}_0 are of type (2, 2).*
- (2) *The 2-class group, $\mathbf{Cl}_2(\mathbb{k})$, of \mathbb{k} is of type (2, 2, 2).*
- (3) *The discriminant of \mathbb{k} is: $\text{disc}(\mathbb{k}) = \text{disc}(k) \cdot \text{disc}(k_0) \cdot \text{disc}(\bar{k}_0) = 2^4 p_1^2 p_2^2 q^2$.*
- (4) *\mathbb{k} has seven unramified quadratic extensions within $\mathbb{k}_2^{(1)}$. They are given by:*

$$\begin{aligned} \mathbb{K}_1 &= \mathbb{k}(\sqrt{p_1}), & \mathbb{K}_2 &= \mathbb{k}(\sqrt{p_2}), & \mathbb{K}_3 &= \mathbb{k}(\sqrt{q}), \\ \mathbb{K}_4 &= \mathbb{k}(\sqrt{\pi_1\pi_3}), & \mathbb{K}_5 &= \mathbb{k}(\sqrt{\pi_1\pi_4}), & \mathbb{K}_6 &= \mathbb{k}(\sqrt{\pi_2\pi_3}) \text{ and } \mathbb{K}_7 = \mathbb{k}(\sqrt{\pi_2\pi_4}). \end{aligned}$$

- (5) *$\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$ are intermediate fields between \mathbb{k} and its genus field $\mathbb{k}^{(*)}$. The fields $\mathbb{K}_4 \simeq \mathbb{K}_7$ and $\mathbb{K}_5 \simeq \mathbb{K}_6$ are pairwise conjugate and thus isomorphic. Consequently $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$ are absolutely abelian, whereas $\mathbb{K}_4, \mathbb{K}_5, \mathbb{K}_6, \mathbb{K}_7$ are absolutely non-normal over \mathbb{Q} .*

- (6) \mathbb{k} has seven unramified bicyclic biquadratic extensions within $\mathbb{k}_2^{(1)}$. One of them is

$$\mathbb{L}_1 = \mathbb{K}_1.\mathbb{K}_2.\mathbb{K}_3 = \mathbb{k}^{(*)} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{q}, \sqrt{-1}),$$

the absolute genus field of \mathbb{k} and the others are given by:

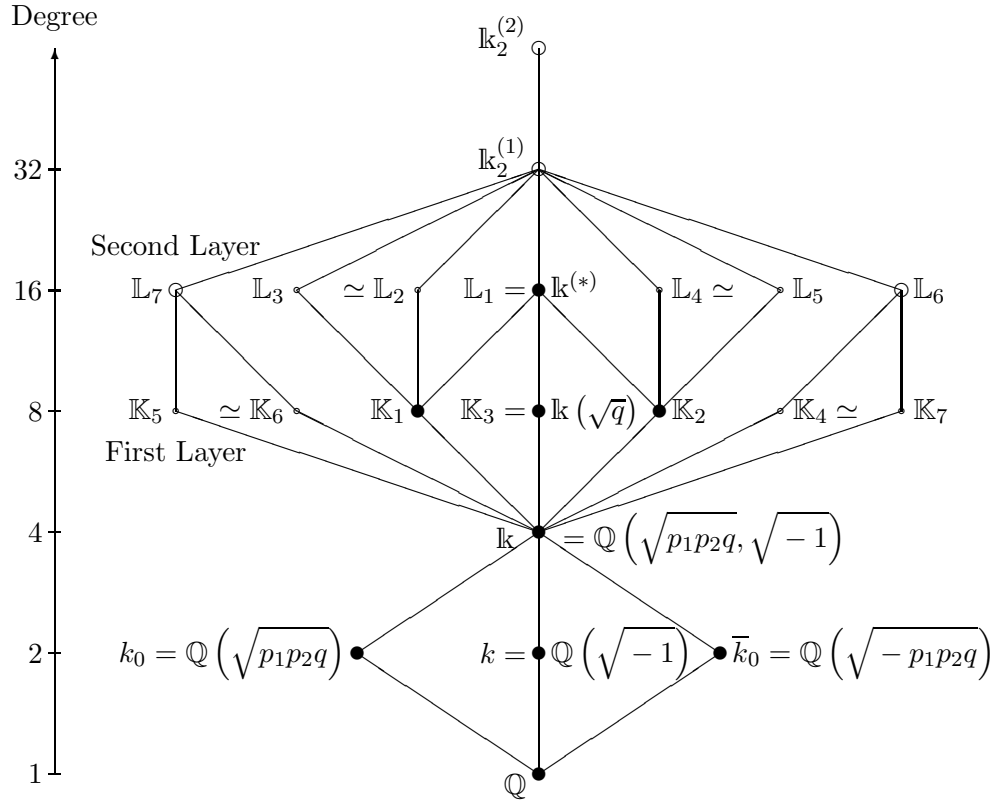
$$\mathbb{L}_2 = \mathbb{K}_1.\mathbb{K}_4.\mathbb{K}_6, \quad \mathbb{L}_3 = \mathbb{K}_1.\mathbb{K}_5.\mathbb{K}_7, \quad \mathbb{L}_4 = \mathbb{K}_2.\mathbb{K}_4.\mathbb{K}_5 \text{ and } \mathbb{L}_5 = \mathbb{K}_2.\mathbb{K}_6.\mathbb{K}_7.$$

Moreover $\mathbb{L}_2 \simeq \mathbb{L}_3$ and $\mathbb{L}_4 \simeq \mathbb{L}_5$ are pairwise conjugate and thus isomorphic and absolutely non-normal, and

$$\mathbb{L}_6 = \mathbb{K}_3.\mathbb{K}_4.\mathbb{K}_7 \text{ and } \mathbb{L}_7 = \mathbb{K}_3.\mathbb{K}_5.\mathbb{K}_6$$

are absolutely Galois over \mathbb{Q} .

FIGURE 1. Subfield lattice of the Hilbert 2-class field $\mathbb{k}_2^{(1)}$ of \mathbb{k}



2.2. Structure of $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$. Let \mathcal{H}_1 (resp. $\mathcal{H}_2, \mathcal{H}_3$) be the prime ideal of \mathbb{k} above π_1 (resp. π_2, π_3).

Theorem 2. *Keep the preceding assumptions, then*

$$(1) \mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle \simeq (2, 2, 2).$$

$$(2) \mathbf{Cl}_2(\mathbb{K}_3) \simeq \begin{cases} (2^{n+1}, 2^{m+1}) & \text{if } \left(\frac{p_1}{p_2}\right) = -1, \\ (2^{n+2}, 2^m) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \end{cases}$$

where n and m are determined by:

$$2^{m+1} = h(-p_1 p_2), \quad m \geq 2, \quad \text{and } 2^n = h(p_1 p_2), \quad n \geq 1.$$

(3) *The length of the 2-class field tower of \mathbb{k} is 2.*

(4) *In dependence on the sign of $N(\varepsilon_{p_1 p_2})$, the second 2-class group $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of \mathbb{k} is given by:*

(i) *If $N(\varepsilon_{p_1 p_2}) = -1$, then*

$$G = \langle \rho, \tau, \sigma : \quad \rho^4 = \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1, \quad \sigma^{2^m} = \tau^{2^{n+1}}, \quad \rho^2 = \tau^{2^n} \sigma^{2^{m-1}}, \\ [\tau, \sigma] = 1, \quad [\sigma, \rho] = \sigma^{2^m-2}, \quad [\rho, \tau] = \tau^2 \rangle.$$

(ii) *If $N(\varepsilon_{p_1 p_2}) = 1$, then*

$$G = \langle \rho, \tau, \sigma : \quad \rho^4 = \sigma^{2^m} = \tau^{2^{n+2}} = 1, \quad \rho^2 = \tau^{2^{n+1}} \sigma^{2^{m-1}} \text{ or } \rho^2 = \sigma^{2^{m-1}}, \\ [\tau, \sigma] = 1, \quad [\rho, \sigma] = \sigma^2, \quad [\tau, \rho] = \tau^{2^{n+1}-2} \rangle.$$

(5) *The derived subgroup of G is $G' = \langle \sigma^2, \tau^2 \rangle \simeq \mathbf{Cl}_2(\mathbb{k}_2^{(1)})$. It is of type*

$$\begin{cases} (2^{\min(n, m-1)}, 2^{\max(m, n+1)}) = (2, 2^{\max(m, n+1)}) & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ (2^{n+1}, 2^{m-1}) & \text{if } N(\varepsilon_{p_1 p_2}) = 1. \end{cases}$$

(6) *The coclass of G is equal to 4 if $\left(\frac{p_1}{p_2}\right) = 1$, $N(\varepsilon_{p_1 p_2}) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, and it is equal to 3 otherwise. The nilpotency class of G is given by:*

$$\begin{cases} m+1 & \text{if } \left(\frac{p_1}{p_2}\right) = -1, \\ m & \text{if } \left(\frac{p_1}{p_2}\right) = 1 \text{ and } \left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1, \\ n+2 & \text{if } \left(\frac{p_1}{p_2}\right) = 1 \text{ and } \begin{cases} N(\varepsilon_{p_1 p_2}) = -1 \text{ or} \\ N(\varepsilon_{p_1 p_2}) = 1 \text{ and } \left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1. \end{cases} \end{cases}$$

2.3. Abelian type invariants and capitulation kernels. Let N_j denote the subgroup $N_{\mathbb{K}_j/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_j))$ of $\mathbf{Cl}_2(\mathbb{k})$ and let $\kappa_{\mathbb{K}/\mathbb{k}}$ denote the (principalization- or capitulation-)kernel of the natural class extension homomorphism $J_{\mathbb{K}/\mathbb{k}} : \mathbf{Cl}_2(\mathbb{k}) \rightarrow \mathbf{Cl}_2(\mathbb{K})$, where \mathbb{K} is an unramified extension of \mathbb{k} within $\mathbb{k}_2^{(1)}$.

Theorem 3. *Put $\pi = \left(\frac{\pi_1}{\pi_3}\right)$.*

(1) *For all $j \neq 3$, there are exactly four classes which capitulate in \mathbb{K}_j , but only two classes which capitulate in \mathbb{K}_3 . More precisely, we have:*

$$(i) \quad \kappa_{\mathbb{K}_1/\mathbb{k}} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle, \quad \kappa_{\mathbb{K}_2/\mathbb{k}} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3] \rangle,$$

$$\kappa_{\mathbb{K}_3/\mathbb{k}} = \begin{cases} \langle [\mathcal{H}_1\mathcal{H}_2] \rangle & \text{if } N(\varepsilon_{p_1p_1}) = 1, \\ \langle [\mathcal{H}_1\mathcal{H}_3] \rangle \text{ or } \langle [\mathcal{H}_2\mathcal{H}_3] \rangle & \text{if } N(\varepsilon_{p_1p_1}) = -1. \end{cases}$$

Moreover, we have:

$$\begin{cases} \kappa_{\mathbb{K}_1/\mathbb{k}} = N_2(\mathbf{Cl}_2(\mathbb{K}_2)), & \kappa_{\mathbb{K}_2/\mathbb{k}} = N_1(\mathbf{Cl}_2(\mathbb{K}_1)) \text{ if } \left(\frac{p_1}{p_2}\right) = 1, \\ \kappa_{\mathbb{K}_1/\mathbb{k}} = N_1(\mathbf{Cl}_2(\mathbb{K}_1)), & \kappa_{\mathbb{K}_2/\mathbb{k}} = N_2(\mathbf{Cl}_2(\mathbb{K}_2)) \text{ if } \left(\frac{p_1}{p_2}\right) = -1. \end{cases}$$

$$(ii) \quad \kappa_{\mathbb{K}_4/\mathbb{k}} = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle, \quad \kappa_{\mathbb{K}_5/\mathbb{k}} = \langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle, \quad \kappa_{\mathbb{K}_6/\mathbb{k}} = \langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$$

and $\kappa_{\mathbb{K}_7/\mathbb{k}} = \langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$. Moreover, we have:

(a) If $\left(\frac{p_1}{p_2}\right) = 1$, then

$$\begin{aligned} \kappa_{\mathbb{K}_4/\mathbb{k}} &= \begin{cases} N_4(\mathbf{Cl}_2(\mathbb{K}_4)) & \text{if } \pi = 1, \\ N_7(\mathbf{Cl}_2(\mathbb{K}_7)) & \text{if } \pi = -1, \end{cases} & \kappa_{\mathbb{K}_5/\mathbb{k}} &= \begin{cases} N_5(\mathbf{Cl}_2(\mathbb{K}_5)) & \text{if } \pi = 1, \\ N_6(\mathbf{Cl}_2(\mathbb{K}_6)) & \text{if } \pi = -1, \end{cases} \\ \kappa_{\mathbb{K}_6/\mathbb{k}} &= \begin{cases} N_6(\mathbf{Cl}_2(\mathbb{K}_6)) & \text{if } \pi = 1, \\ N_5(\mathbf{Cl}_2(\mathbb{K}_5)) & \text{if } \pi = -1, \end{cases} & \kappa_{\mathbb{K}_7/\mathbb{k}} &= \begin{cases} N_7(\mathbf{Cl}_2(\mathbb{K}_7)) & \text{if } \pi = 1, \\ N_4(\mathbf{Cl}_2(\mathbb{K}_4)) & \text{if } \pi = -1. \end{cases} \end{aligned}$$

(b) If $\left(\frac{p_1}{p_2}\right) = -1$, then

$$\begin{aligned} \kappa_{\mathbb{K}_4/\mathbb{k}} &= \begin{cases} N_4(\mathbf{Cl}_2(\mathbb{K}_4)) & \text{if } \pi = -1, \\ N_7(\mathbf{Cl}_2(\mathbb{K}_7)) & \text{if } \pi = 1, \end{cases} & \kappa_{\mathbb{K}_5/\mathbb{k}} &= \begin{cases} N_6(\mathbf{Cl}_2(\mathbb{K}_6)) & \text{if } \pi = -1, \\ N_5(\mathbf{Cl}_2(\mathbb{K}_5)) & \text{if } \pi = 1, \end{cases} \\ \kappa_{\mathbb{K}_6/\mathbb{k}} &= \begin{cases} N_5(\mathbf{Cl}_2(\mathbb{K}_5)) & \text{if } \pi = -1, \\ N_6(\mathbf{Cl}_2(\mathbb{K}_6)) & \text{if } \pi = 1, \end{cases} & \kappa_{\mathbb{K}_7/\mathbb{k}} &= \begin{cases} N_7(\mathbf{Cl}_2(\mathbb{K}_7)) & \text{if } \pi = -1, \\ N_4(\mathbf{Cl}_2(\mathbb{K}_4)) & \text{if } \pi = 1. \end{cases} \end{aligned}$$

(2) All the extensions \mathbb{K}_j satisfy Taussky's condition (A), i.e. $\kappa_{\mathbb{K}_j/\mathbb{k}} \cap N_j > 1$ ([26]).

(3) For all j , $\kappa_{\mathbb{L}_j/\mathbb{k}} = \mathbf{Cl}_2(\mathbb{k})$ (total 2-capitulation), and each \mathbb{L}_j is of type (A).

Theorem 4. Let $2^n = h(p_1p_2)$, $2^{m+1} = h(-p_1p_2)$, where $n \geq 1$ and $m \geq 2$. Put $\beta = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right)$.

(1) The abelian type invariants of the 2-class groups $\mathbf{Cl}_2(\mathbb{K}_j)$ are given by:

- (i) $\mathbf{Cl}_2(\mathbb{K}_1)$ and $\mathbf{Cl}_2(\mathbb{K}_2)$ are of type $(2, 2, 2)$ if $\left(\frac{p_1}{p_2}\right) = 1$ and of type $(2, 4)$ otherwise.
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{K}_4)$, $\mathbf{Cl}_2(\mathbb{K}_5)$, $\mathbf{Cl}_2(\mathbb{K}_6)$ and $\mathbf{Cl}_2(\mathbb{K}_7)$ are of type $(2, 2, 2)$ if $\left(\frac{\pi_1}{\pi_3}\right) = -1$, and of type $(2, 4)$ otherwise.

(iii) If $\left(\frac{p_1}{p_2}\right) = -1$, then we have:

$$\text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{K}_4) \simeq \mathbf{Cl}_2(\mathbb{K}_7) \simeq (2, 4), \\ \mathbf{Cl}_2(\mathbb{K}_5) \simeq \mathbf{Cl}_2(\mathbb{K}_6) \simeq (2, 2, 2); \end{cases}$$

$$\text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{K}_4) \simeq \mathbf{Cl}_2(\mathbb{K}_7) \simeq (2, 2, 2), \\ \mathbf{Cl}_2(\mathbb{K}_5) \simeq \mathbf{Cl}_2(\mathbb{K}_6) \simeq (2, 4). \end{cases}$$

(2) The abelian type invariants of the 2-class groups $\mathbf{Cl}_2(\mathbb{L}_j)$ are given by:

- (i) $\mathbf{Cl}_2(\mathbb{L}_1) \simeq \begin{cases} (2^{\min(m,n)}, 2^{\max(m+1,n+1)}) & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ (2^m, 2^{n+1}) & \text{if } N(\varepsilon_{p_1 p_2}) = 1. \end{cases}$
- (ii) If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = \left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{L}_2)$, $\mathbf{Cl}_2(\mathbb{L}_3)$, $\mathbf{Cl}_2(\mathbb{L}_4)$ and $\mathbf{Cl}_2(\mathbb{L}_5)$ are of type (2, 4).
If $\left(\frac{p_1}{p_2}\right) = -\left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{L}_2)$, $\mathbf{Cl}_2(\mathbb{L}_3)$, $\mathbf{Cl}_2(\mathbb{L}_4)$ and $\mathbf{Cl}_2(\mathbb{L}_5)$ are of type (2, 2, 2).
- (iii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{L}_6)$ and $\mathbf{Cl}_2(\mathbb{L}_7)$ are of type $(2, 2^{n+2})$ if $\left(\frac{\pi_1}{\pi_3}\right) = 1$, otherwise we have:
- $$\mathbf{Cl}_2(\mathbb{L}_6) \simeq \begin{cases} (2^{m-1}, 2^{n+2}) & \text{if } \beta = 1, \\ (2^{\min(m-1,n+1)}, 2^{\max(m,n+2)}) & \text{if } \beta = -1, \end{cases}$$
- $$\mathbf{Cl}_2(\mathbb{L}_7) \simeq \begin{cases} (2^{\min(m-1,n+1)}, 2^{\max(m,n+2)}) & \text{if } \beta = 1, \\ (2^{m-1}, 2^{n+2}) & \text{if } \beta = -1, \end{cases}$$
- If $\left(\frac{p_1}{p_2}\right) = -1$, then
- $$\mathbf{Cl}_2(\mathbb{L}_6) \simeq \begin{cases} (2^{n+1}, 2^m) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \\ (2^{\min(m-1,n)}, 2^{\max(m+1,n+2)}) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \end{cases}$$
- and
- $$\mathbf{Cl}_2(\mathbb{L}_7) \simeq \begin{cases} (2^{\min(m-1,n)}, 2^{\max(m+1,n+2)}) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \\ (2^{n+1}, 2^m) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \end{cases}$$

3. PRELIMINARY RESULTS

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the conditions (1). Put $k_0 = \mathbb{Q}(\sqrt{p_1 p_2 q})$, $\bar{k}_0 = \mathbb{Q}(\sqrt{-p_1 p_2 q})$, $k_1 = \mathbb{Q}(\sqrt{p_1 p_2})$, $\bar{k}_1 = \mathbb{Q}(\sqrt{-p_1 p_2})$, $\varepsilon_{p_1 p_2 q} = x + y\sqrt{p_1 p_2 q}$ and $\varepsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$. Let $\left(\frac{g, h}{p}\right)$ denote the quadratic Hilbert symbol for some prime p .

Lemma 1. *Keep the notations above. Then*

- (1) *If $N(\varepsilon_{p_1 p_2}) = 1$, then $2p_1(a \pm 1)$ i.e. $2p_2(a \mp 1)$ is a square in \mathbb{N} .*
- (2) *$p_1 p_2(x \pm 1)$ i.e. $q(x \mp 1)$ is a square in \mathbb{N} .*

Proof. (1) If $N(\varepsilon_{p_1 p_2}) = 1$, then $\left(\frac{p_1}{p_2}\right) = 1$ and $a^2 - 1 = b^2 p_1 p_2$. Therefore, according to [2, Lemma 5, p. 386] and the decomposition uniqueness in \mathbb{Z} , there are three possible cases: $a \pm 1$ or $p_1(a \pm 1)$ or $2p_1(a \pm 1)$ is a square in \mathbb{N} :

- (a) If $a \pm 1$ is a square in \mathbb{N} , then $\left(\frac{2}{p_1}\right) = -1$, which is false.
- (b) If $p_1(a \pm 1)$ is a square in \mathbb{N} , then $\left(\frac{p_1}{p_2}\right) = \left(\frac{2}{p_1}\right) = -1$, which is false. Thus the result.

(2) As $N(\varepsilon_{p_1 p_2 q}) = 1$, then $x^2 - 1 = y^2 p_1 p_2 q$. Proceeding as in (1) we get that the only possible case is:
$$\begin{cases} x \pm 1 = p_1 p_2 y_2^2, \\ x \mp 1 = q y_1^2, \end{cases} \quad \text{hence the result.} \quad \square$$

Lemma 2. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the conditions (1). Put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$, $\mathbb{K}_3^+ = \mathbb{Q}(\sqrt{q}, \sqrt{p_1 p_2})$, $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1 p_2}, i)$ and $F = \mathbb{Q}(\sqrt{-q}, \sqrt{-p_1 p_2})$. Then*

- (1) $\{\varepsilon_{p_1 p_2 q}\}$ is a FSU of both \mathbb{k} and F .
- (2) The FSU's of \mathbb{K}_3^+ and \mathbb{K}_3 are $\{\varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}\}$ and $\{\varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}, \sqrt{i \varepsilon_q}\}$ respectively.
- (3) $q(F/\mathbb{Q}) = 1$, $q(\mathbb{K}_3^+/\mathbb{Q}) = 2$ and $q(\mathbb{K}_3/\mathbb{Q}) = 4$.
- (4) $h(F) = 2h(-p_1 p_2)$, $h(\mathbb{K}_3^+) = 2h(p_1 p_2)$ and $h(\mathbb{K}_3) = 2h(p_1 p_2)h(-p_1 p_2)$.

Proof. Determine first the FSU of \mathbb{K}_3^+ . If $N(\varepsilon_{p_1 p_2}) = -1$, then only ε_q , $\varepsilon_{p_1 p_2 q}$ and $\varepsilon_q \varepsilon_{p_1 p_2 q}$ can be squares in \mathbb{K}_3^+ .

According to [3, Lemma 3, p. 2199], we get ε_q is not a square in $\mathbb{Q}(\sqrt{q})$; but $2\varepsilon_q$ is. On the other hand, from Lemma 1, we have $p_1 p_2(x \pm 1)$ is a square in \mathbb{N} , thus $\sqrt{2\varepsilon_{p_1 p_2 q}} = y_1 \sqrt{q} + y_2 \sqrt{p_1 p_2}$, with some $y_j \in \mathbb{Z}$. This yields that $\varepsilon_{p_1 p_2 q}$ and $2\varepsilon_{p_1 p_2 q}$ are not squares in $\mathbb{Q}(\sqrt{p_1 p_2 q})$; and $2\varepsilon_{p_1 p_2 q}$ is in \mathbb{K}_3^+ . Therefore $\varepsilon_q \varepsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , which implies that $\{\varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}\}$ is a FSU of \mathbb{K}_3^+ , and thus $q(\mathbb{K}_3^+/\mathbb{Q}) = 2$. Moreover, as $2\varepsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$, then [4, proposition 3, p. 112] implies that $\{\varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}, \sqrt{i \varepsilon_q}\}$ is a FSU of \mathbb{K}_3 , and thus $q(\mathbb{K}_3/\mathbb{Q}) = 4$. We find the same results if we assume that $N(\varepsilon_{p_1 p_2}) = 1$.

We know, from Lemma 1, that $x \pm 1$ is not a square in \mathbb{N} ; hence [4, Applications 1 (i), p. 114] implies that $\{\varepsilon_{p_1 p_2 q}\}$ is a FSU of \mathbb{k} .

For the field $F = \mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-q})$, we know, according to [4, Applications 2, p. 114], that $\{\sqrt{-\varepsilon_{p_1 p_2 q}}\}$ is a FSU of F if and only if $2q(x \pm 1)$ i.e. $2p_1 p_2(x \pm 1)$ is a square in \mathbb{N} , which is not the case (Lemma 1). Hence $\{\varepsilon_{p_1 p_2 q}\}$ is a FSU of F , and thus $q(F/\mathbb{Q}) = 1$.

Finally, under our assumptions, P. Kaplan states in [14] that $h(p_1 p_2 q) = h(-p_1 p_2 q) = 4$. Therefore, the number class formula implies that $h(\mathbb{K}_3^+) = 2h(p_1 p_2)$, $h(F) = 2h(-p_1 p_2)$ and $h(\mathbb{K}_3) = 2h(p_1 p_2)h(-p_1 p_2)$. \square

Lemma 3. *If $\left(\frac{p_1}{p_2}\right) = 1$, then $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right)$.*

Proof. From [14] we get $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{p_1}{ac+bd}\right)$, where $p_1 = a^2 + b^2$ and $p_2 = c^2 + d^2$; on the other hand, according to [19] we have $\left(\frac{p_1}{ac+bd}\right) = \left(\frac{\pi_1}{\pi_3}\right)$, which implies the result. \square

The following results are deduced from [15].

Theorem 5. *Let $p_1 \equiv p_2 \equiv 5 \pmod{8}$ be different primes and put $F_1 = \mathbb{Q}(\sqrt{p_1 p_2}, i)$.*

- (1) $\mathbf{Cl}_2(\bar{k}_1)$ is of type $(2, 2^m)$, $m \geq 2$. It is generated by $2 = (2, 1 + \sqrt{-p_1 p_2})$, the prime ideal of \bar{k}_1 above 2, and an ideal I of \bar{k}_1 of order 2^m . Moreover

$$\begin{cases} I^{2^{m-1}} \sim \mathfrak{p}_1 & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ I^{2^{m-1}} \sim 2\mathfrak{p}_1 & \text{if } \left(\frac{p_1}{p_2}\right) = -1; \end{cases}$$

where $\mathfrak{p}_1 = (p_1, \sqrt{-p_1 p_2})$ is the prime ideal of \bar{k}_1 above p_1 .

- (2) $\mathbf{Cl}_2(k_1)$ is of type (2^n) , $n \geq 1$, and it is generated by 2_1 , a prime ideal of k_1 above 2.
- (3) $\mathbf{Cl}_2(F_1)$ is of 2-rank equal to 2. It is generated by I and 2_{F_1} , where 2_{F_1} is a prime ideal of F_1 above 2.
- (4) If $\left(\frac{p_1}{p_2}\right) = -1$, then $\mathbf{Cl}_2(F_1) \simeq (2^n, 2^m)$; and, in $\mathbf{Cl}_2(F_1)$, $I^{2^{m-1}} \sim 2_{F_1}^{2^n} \sim \mathfrak{p}_1 \not\sim 1$.
- (5) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then

$$\mathbf{Cl}_2(F_1) \simeq (2^{\min(n, m-1)}, 2^{\max(m-1, n+1)})$$

and $I^{2^{m-1}} \sim 2_{F_1}^{2^n} \sim \mathfrak{p}_1 \not\sim 1$.

- (6) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\mathbf{Cl}_2(F_1) \simeq (2^{n+1}, 2^{m-1})$; moreover $I^{2^{m-1}} \sim 2_{F_1}^{2^{n+1}} \sim \mathfrak{p}_1 \sim 1$.

Using the above theorem, we prove the following lemma.

Lemma 4. *Let $\mathfrak{p}_1 \mathcal{O}_{F_1} = \mathcal{P}_1 \mathcal{P}_2$ and $p_2 \mathcal{O}_{F_1} = \mathcal{P}_3^2 \mathcal{P}_4^2$, then in $\mathbf{Cl}_2(F_1)$ we have:*

- (i) *If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$.*

- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\mathcal{P}_1 \sim z_{F_1}^{2^n} I^{2^{m-2}}$ or $\mathcal{P}_1 \sim I^{2^{m-2}}$.
Moreover $\mathcal{P}_1 \mathcal{P}_3 \sim z_{F_1}^{2^n}$.

Proof. Let $p_1 \mathcal{O}_{\mathbb{Q}(i)} = \pi_1 \pi_2$, $p_2 \mathcal{O}_{\mathbb{Q}(i)} = \pi_3 \pi_4$, $\mathfrak{p}_1 \mathcal{O}_{F_1} = \mathcal{P}_1 \mathcal{P}_2$ and $\mathfrak{p}_2 \mathcal{O}_{F_1} = \mathcal{P}_3 \mathcal{P}_4$, where \mathfrak{p}_2 is the prime ideal of \bar{k}_1 above p_2 , then $(\pi_i) = \mathcal{P}_i^2$, for all i . So, according to [7, Proposition 1], \mathcal{P}_i are not principals in F_1 and they are of order two. On the other hand, as the 2-rank of $\mathbf{Cl}_2(F_1)$ is 2, thus $\mathcal{P}_i \in \langle [2_{F_1}], [I] \rangle$.

(i) In this case, we have $\mathfrak{p}_1 \not\sim 1$, hence $\mathcal{P}_1 \not\sim \mathcal{P}_2$; note that the elements of order two in $\mathbf{Cl}_2(F_1)$ are $z_{F_1}^{2^{n-1}} I^{2^{m-2}}$, $z_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ and $z_{F_1}^{2^n} \sim I^{2^{m-1}}$. Therefore \mathcal{P}_1 is equivalent to one of these three elements. As $\mathcal{P}_1 \sim z_{F_1}^{2^n} \sim I^{2^{m-1}}$ can not occur, if not we would have, by applying the norm N_{F_1/\bar{k}_1} , $\mathfrak{p}_1 \sim I^{2^m} \sim 1$, which is false. Thus $\mathcal{P}_1 \sim z_{F_1}^{2^{n-1}} I^{2^{m-2}}$ and $\mathcal{P}_2 \sim z_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ or $\mathcal{P}_1 \sim z_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ and $\mathcal{P}_2 \sim z_{F_1}^{2^{n-1}} I^{2^{m-2}}$. Hence with out loss of generality we can choose $\mathcal{P}_1 \sim z_{F_1}^{2^{n-1}} I^{2^{m-2}}$.

(ii) In this case, we have $\mathfrak{p}_1 \sim \mathfrak{p}_2 \sim 1$, hence $\mathcal{P}_1 \sim \mathcal{P}_2$ and $\mathcal{P}_3 \sim \mathcal{P}_4$. On the other hand, according to [7, Proposition 1], $\mathcal{P}_1 \mathcal{P}_3$ is not principal in F_1 . To this end, note that the elements of order two in $\mathbf{Cl}_2(F_1)$ are $z_{F_1}^{2^n} I^{2^{m-2}}$, $z_{F_1}^{2^n} I^{-2^{m-2}}$ and $I^{-2^{m-2}}$. Therefore \mathcal{P}_1 is equivalent to one of these three elements. As $\mathcal{P}_1 \sim z_{F_1}^{2^n}$ can not occur, as otherwise, by applying the norm N_{F_1/\bar{k}_1} , we get $\mathfrak{p}_1 \sim z^{2^n} \sim 1$, which is false. Thus $\mathcal{P}_1 \sim I^{2^{m-2}}$ and $\mathcal{P}_3 \sim z_{F_1}^{2^n} I^{2^{m-2}}$ or $\mathcal{P}_1 \sim z_{F_1}^{2^{n-1}} I^{2^{m-2}}$ and $\mathcal{P}_3 \sim I^{2^{m-2}}$. Hence $\mathcal{P}_1 \mathcal{P}_3 \sim z_{F_1}^{2^n}$. \square

The following lemma gives some relations between $N(\varepsilon_{p_1 p_2})$ and the positive integers n, m . It is a deduction from [14] and [25].

Lemma 5. *Let $p_1 \equiv p_2 \equiv 5 \pmod{8}$ be different primes.*

(1) *Suppose that $N(\varepsilon_{p_1 p_2}) = -1$, then*

(i) *If $\left(\frac{p_1}{p_2}\right) = -1$, then $n = 1$ and $m \geq 2$. Moreover:*

$$(a) \ m \geq 3 \Leftrightarrow \left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = 1.$$

$$(b) \ m = 2 \Leftrightarrow \left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = -1.$$

(ii) *If $\left(\frac{p_1}{p_2}\right) = 1$, then $n \geq 2$ and $m = 2$. Moreover:*

$$(a) \ \text{If } \left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = -1, \text{ then } n = 2.$$

$$(b) \ \text{If } \left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1, \text{ then } n \geq 2.$$

(2) *Suppose that $N(\varepsilon_{p_1 p_2}) = 1$, then $\left(\frac{p_1}{p_2}\right) = 1$, $n \geq 1$ and $m \geq 2$. Moreover:*

- (i) If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then $n = 1$ and $m \geq 3$.
- (ii) If $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$, then $m = 2$ and $n \geq 2$.

4. ON THE 2-CLASS FIELD TOWER OF \mathbb{k}

In this section, we will prove that the length of the 2-class field tower of \mathbb{k} is 2. Show first the following lemma:

Lemma 6. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the conditions (1). Put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$, $\mathbb{K}_3^+ = \mathbb{Q}(\sqrt{q}, \sqrt{p_1 p_2})$ and $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1 p_2}, i)$. Then*

- (1) *The rank of $\text{Cl}_2(\mathbb{K}_3)$ is equal to 2.*
- (2) *The 2-class group of \mathbb{K}_3^+ is cyclic.*

Proof. (1) Put $F_2 = \mathbb{Q}(\sqrt{q}, i)$ and let $\varepsilon_q = 1 + \sqrt{q}$ denote the fundamental unit of $\mathbb{Q}(\sqrt{q})$. Then, according to [3], $2\varepsilon_q$ is a square in F_2^+ , thus [4] yields that the unit group of F_2 is $E_{F_2} = \langle i, \sqrt{i\varepsilon_q} \rangle$. As the class number of F_2 is odd, then the 2-rank of \mathbb{K}_3 is: $r = t - e - 1$, where t is the number of ramified primes (finite and infinite) in \mathbb{K}_3/F_2 and $2^e = [E_{F_2} : E_{F_2} \cap N_{\mathbb{K}_3/F_2}(\mathbb{K}_3^\times)]$. The following diagram helps us to calculate t .

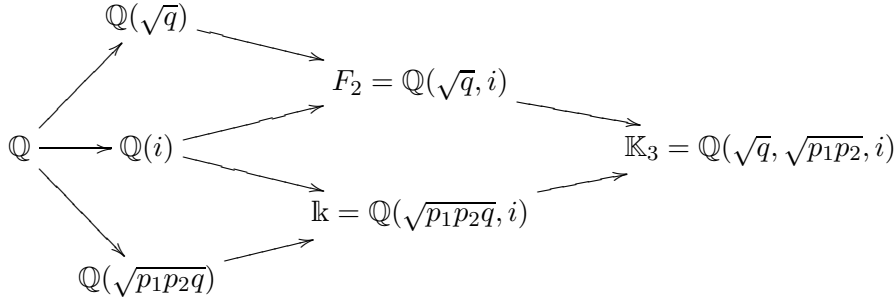


FIGURE 2. Primes ramifying in \mathbb{K}_3/F_2

Let p be a prime, we denote by \mathfrak{p}_M a prime ideal of the extension M/\mathbb{Q} lying above p and $e(\mathfrak{p}_M/p)$ its ramification index. As the extension \mathbb{K}_3/\mathbb{k} is unramified (see [6]), then $e(\mathfrak{p}_{F_2}/p) \cdot e(\mathfrak{p}_{\mathbb{K}_3}/\mathfrak{p}_{F_2}) = e(\mathfrak{p}_{\mathbb{k}}/p)$. Moreover, 2 and q are ramified in F_2 and \mathbb{k} , then $e(2_{\mathbb{K}_3}/2_{F_2}) = 1$ and $e(q_{\mathbb{K}_3}/q_{F_2}) = 1$. On the other hand, for all $j \in \{1, 2\}$ $\left(\frac{p_j}{q}\right) = -1$, hence $e(\mathfrak{p}_{j, F_2}/p_j) = 1$, and since $e(\mathfrak{p}_{j, \mathbb{k}}/p) = 2$ so

$e(\mathfrak{p}_{j, \mathbb{K}_3}/\mathfrak{p}_{j, F_2}) = 2$. Thus $t = 4$ and $r = 3 - e$. To calculate the number e , we have to find the units of F_2 which are norms of elements of \mathbb{K}_3^\times/F_2 . Let \mathfrak{p} be a prime ideal of F_2 , then by Hilbert symbol properties and according to [12, p. 205], we have:

- If \mathfrak{p} is not above p_1 and p_2 , then $v_{\mathfrak{p}}(\sqrt{i\varepsilon_q}) = v_{\mathfrak{p}}(p_1p_2) = v_{\mathfrak{p}}(i) = 0$, hence
$$\left(\frac{p_1p_2, \sqrt{i\varepsilon_q}}{\mathfrak{p}}\right) = \left(\frac{p_1p_2, i}{\mathfrak{p}}\right) = 1.$$
- If \mathfrak{p} lies above p_1 or p_2 , then $v_{\mathfrak{p}}(\sqrt{i\varepsilon_q}) = v_{\mathfrak{p}}(i) = 0$ and $v_{\mathfrak{p}}(p_1p_2) = 1$, thus
$$\left(\frac{p_1p_2, i}{\mathfrak{p}}\right) = \left(\frac{i}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(p_1p_2)} = \left(\frac{i}{\mathfrak{p}}\right) = \left(\frac{-1}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) = \left(\frac{-1}{p_j}\right) = 1, \text{ where } j \in \{1, 2\}.$$
$$\begin{aligned} \left(\frac{p_1p_2, \sqrt{i\varepsilon_q}}{\mathfrak{p}}\right) &= \left(\frac{p_1p_2, 2}{\mathfrak{p}}\right) \left(\frac{p_1p_2, 1+i}{\mathfrak{p}}\right) \left(\frac{p_1p_2, \sqrt{2\varepsilon_q}}{\mathfrak{p}}\right) \\ &= \left(\frac{p_1p_2, i}{\mathfrak{p}}\right) \left(\frac{p_1p_2, 1+i}{\mathfrak{p}}\right) \left(\frac{p_1p_2, \sqrt{2\varepsilon_q}}{\mathfrak{p}}\right) \\ &= \left(\frac{p_1p_2, 1+i}{\mathfrak{p}}\right) \left(\frac{p_1p_2, \sqrt{2\varepsilon_q}}{\mathfrak{p}}\right) \\ &= 1. \left(\frac{2}{p_j}\right), \text{ where } j \in \{1, 2\} \\ &= -1. \end{aligned}$$

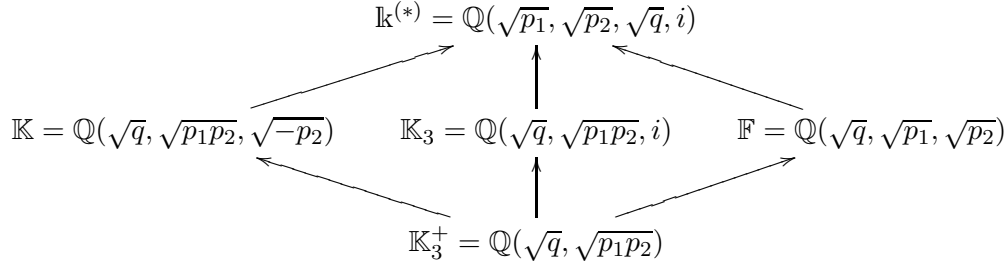
Consequently, $e = 1$, and thus $r = 2$.

Proceeding similarly, we prove that the 2-class group of \mathbb{K}_3^+ is cyclic. \square

Theorem 6. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the conditions (1). Put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2q}, i)$ and $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$. Denote by $\mathbb{k}_2^{(2)}$ the second Hilbert 2-class field of \mathbb{k} and put $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$, then:*

- (1) *The 2-class field tower of \mathbb{k} stops at $\mathbb{k}_2^{(2)}$.*
- (2) *The order of G satisfies $|G| = 2h(\mathbb{K}_3) \geq 64$.*

Proof. Let $\mathbb{k}^{(*)}$ denote the genus field of \mathbb{k} , then $\mathbb{k}^{(*)}/\mathbb{K}_3^+$ is a V_4 -extension of CM-type fields, The following diagram (Figure 3) clarifies this, put $\mathbb{K} = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, \sqrt{-p_2})$ and $\mathbb{F} = \mathbb{Q}(\sqrt{q}, \sqrt{p_1}, \sqrt{p_2})$:


 FIGURE 3. Subfields of $k^{(*)}/K_3^+$

So, according to [17], we have:

$$h(k^{(*)}) = \frac{Q_{k^{(*)}}}{Q_K Q_{K_3}} \cdot \frac{\omega_{k^{(*)}}}{\omega_K \omega_{K_3}} \cdot \frac{h(K)h(K_3)h(F)}{h(K_3^+)^2}. \quad (2)$$

Note that $\omega_K = 2$ and $\omega_{k^{(*)}} = \omega_{K_3} = 12$ or 4 according as $q = 3$ or not; moreover $W_{k^{(*)}} = W_{K_3} = \langle i \rangle$ if $q \neq 3$ and $W_{k^{(*)}} = W_{K_3} = \langle i\xi \rangle$ if $q = 3$, where ξ is a primitive 6st root of unity. On the first hand, Lemma 2 yields that $E_{K_3} = \langle i, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}, \sqrt{i \varepsilon_q} \rangle$ or $E_{K_3} = \langle i\xi, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}, \sqrt{i \varepsilon_q} \rangle$, according as $q \neq 3$ or not, so $Q_{K_3} = 2$. On the other hand, [17] implies that $Q_{K_3} | Q_{k^{(*)}} [W_{k^{(*)}} : W_{K_3}] = Q_{k^{(*)}}$, hence $Q_{k^{(*)}} = 2$.

At this end, we know that the 2-class group of $K_3^+ = \mathbb{Q}(\sqrt{q}, \sqrt{p_1 p_2})$ is cyclic of order $2h(p_1 p_2)$ (see Lemmas 2 and 6), moreover F is an unramified quadratic extension of K_3^+ , then

$$h(F) = \frac{h(K_3^+)}{2} = h(p_1 p_2).$$

From Lemma 2 we get $E_{K_3^+} = \langle -1, \varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}} \rangle$, so, according to [1, Proposition 3], $E_K = \langle -1, \varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}} \rangle$ or $\langle -1, \varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{-\varepsilon_q \varepsilon_{p_1 p_2 q}} \rangle$. As $2\varepsilon_q$ and $2\varepsilon_{p_1 p_2 q}$ are squares in K_3^+ , so $p_2 \varepsilon_q$ and $p_2 \varepsilon_{p_1 p_2 q}$ are not, if not we obtain that $2p_2$ is a square in K_3^+ , which is false. Similarly, $p_2 \varepsilon_q \varepsilon_{p_1 p_2 q}$ is not square in K_3^+ , since $\varepsilon_q \varepsilon_{p_1 p_2 q}$ is. Furthermore $p_2 \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}$ and $p_2 \varepsilon_q \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}$ are not squares in K_3^+ , if not, by applying the norm $N_{K_3^+/\mathbb{Q}(\sqrt{q})}$, we obtain that ε_q is a square in $\mathbb{Q}(\sqrt{q})$, which is absurd; consequently $E_K = \langle -1, \varepsilon_q, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}} \rangle$, this implies that $q(K/\mathbb{Q}) = 2$. Finally, the class number formula allows us to conclude that

$$h(K) = 4h(p_1 p_2).$$

Hence the equation (2) yields that

$$h(\mathbb{k}^{(*)}) = \frac{h(\mathbb{K}_3)}{2},$$

since $\omega_{\mathbb{K}} = 2$, $W_{\mathbb{K}} = \{-1, 1\}$ and $Q_{\mathbb{K}} = 1$. Moreover, as the 2-rank of $\mathbf{Cl}_2(\mathbb{K}_3)$ is equal to 2 (Lemma 6), so we can apply Proposition 7 of [9], which says that \mathbb{K}_3 has abelian 2-class field tower if and only if it has a quadratic unramified extension $\mathbb{k}^{(*)}/\mathbb{K}_3$ such that $h(\mathbb{k}^{(*)}) = \frac{h(\mathbb{K}_3)}{2}$; therefore \mathbb{K}_3 has abelian 2-class field tower which terminates at the first stage; this implies that the 2-class field tower of \mathbb{k} terminates at $\mathbb{k}_2^{(2)}$, since $\mathbb{k} \subset \mathbb{K}_3$. On the other hand, Lemma 2 yields that $h(\mathbb{K}_3) = 2h(p_1p_2)h(-p_1p_2)$. Moreover, under our conditions, P. Kaplan affirms in [14, Proposition B'_1 , p. 348] that $h(-p_1p_2) \geq 8$, whence $h(\mathbb{K}_3) \geq 32$; which implies that $\mathbb{k}_2^{(1)} \neq \mathbb{k}_2^{(2)}$.

Let us prove now that $|G| = 2h(\mathbb{K}_3) \geq 64$, for this we distinguish the following cases:

Case 1: Assume that $\left(\frac{p_1}{p_2}\right) = -1$, so, according to [14, Proposition B'_1 , p. 348], we have:

- a - If $\left(\frac{p_1p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = -1$, then $h(p_1p_2) = 2$ and $h(-p_1p_2) = 8$, thus $|G| = 64$.
- b - If $\left(\frac{p_1p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = 1$, then $h(p_1p_2) = 2$ and $h(-p_1p_2) \geq 16$, thus $|G| \geq 128$.

Case 2: Assume that $\left(\frac{p_1}{p_2}\right) = 1$, so, according to [14, Proposition B'_4 , p. 349], we have:

- a - If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then [25] implies that $h(p_1p_2) = 2$ and [14] yields $h(-p_1p_2) \geq 16$, thus $|G| \geq 128$.
- b - If $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$, then [14] implies that $h(-p_1p_2) = 8$; moreover $h(p_1p_2) \geq 4$, thus $|G| \geq 128$.
- c - If $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = -1$, then [14] implies that $h(-p_1p_2) = 8$ and [25] yields that $h(p_1p_2) = 4$, thus $|G| = 128$. This ends the proof of the theorem. \square

From the proof of Theorem 6, we deduce the following result:

Corollary 1. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes satisfying the conditions (1). Put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2q}, i)$ and $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$. Denote by $\mathbb{k}_2^{(2)}$ the second Hilbert 2-class field of \mathbb{k} and put $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$, then:*

- (1) $|G| = 64$ if and only if $\left(\frac{p_1}{p_2}\right) = -1$ and $\left(\frac{p_1p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = -1$.

(2) $|G| = 128$ if and only if one of the following conditions holds:

- (i) $\left(\frac{p_1}{p_2}\right) = -1$ and $h(-p_1p_2) = 16$.
- (ii) $\left(\frac{p_1}{p_2}\right) = 1$, $\left(\frac{p_1}{p_2}\right)_4 = -\left(\frac{p_2}{p_1}\right)_4$ and $h(-p_1p_2) = 16$.
- (iii) $\left(\frac{p_1}{p_2}\right) = 1$, $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$ and $h(p_1p_2) = 4$.
- (iv) $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = -1$.

(3) $|G| \geq 256$ if and only if one of the following conditions holds:

- (i) $\left(\frac{p_1}{p_2}\right) = -1$ and $h(-p_1p_2) \geq 32$.
- (ii) $\left(\frac{p_1}{p_2}\right) = 1$, $\left(\frac{p_1}{p_2}\right)_4 = -\left(\frac{p_2}{p_1}\right)_4$ and $h(-p_1p_2) \geq 32$.
- (iii) $\left(\frac{p_1}{p_2}\right) = 1$, $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$ and $h(p_1p_2) \geq 8$.

5. PROOFS OF THE MAIN RESULTS

Recall first the following result from [12, p. 205].

Lemma 7. *If \mathcal{H} is an unramified ideal in some extension $\mathbb{K}/\mathbb{k} = \mathbb{k}(\sqrt{x})/\mathbb{k}$, then the quadratic residue symbol is given by the Artin symbol $\varphi = \left(\frac{\mathbb{k}(\sqrt{x})/\mathbb{k}}{\mathcal{H}}\right)$ as follows: $\left(\frac{x}{\mathcal{H}}\right) = \sqrt{x}^{\varphi-1}$.*

5.1. Proof of Theorem 2. (1) The assertion $\mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle \simeq (2, 2, 2)$ of Theorem 2 is proved in [5] and [7]. In the following pages, we will prove the other assertions.

(2) **Types of $\mathbf{Cl}_2(\mathbb{K}_3)$** To prove the second assertion we will use the techniques that F. Lemmermeyer has used in some of his works see for example [15] or [18]. Consider the following diagram (Figure 3):

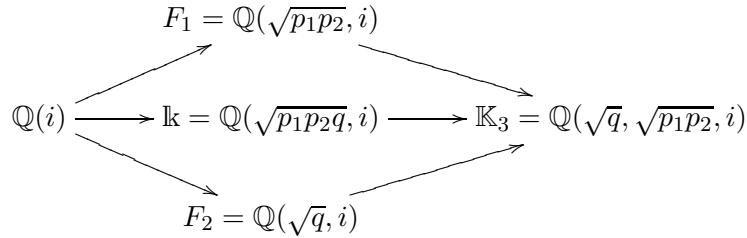


FIGURE 4. Subfields of $\mathbb{K}_3/\mathbb{Q}(i)$

Note first that \mathcal{H}_j , \mathcal{P}_j coincide and remain inert in \mathbb{K}_3 , and that $\mathfrak{p}_1 \mathcal{O}_{\mathbb{K}_3} = \mathcal{H}_1 \mathcal{H}_2 \mathcal{O}_{\mathbb{K}_3} = \mathcal{P}_1 \mathcal{P}_2 \mathcal{O}_{\mathbb{K}_3}$, $\mathfrak{p}_2 \mathcal{O}_{\mathbb{K}_3} = \mathcal{H}_3 \mathcal{H}_4 \mathcal{O}_{\mathbb{K}_3} = \mathcal{P}_3 \mathcal{P}_4 \mathcal{O}_{\mathbb{K}_3}$, where \mathfrak{p}_2 is the prime ideal of k_1 above p_2 . On the other hand, as \mathcal{H}_j , with $j \in \{1, 2, 3, 4\}$, is unramified

in $\mathbb{K}_3/\mathbb{k} = \mathbb{k}(\sqrt{q})/\mathbb{k} = \mathbb{k}(\sqrt{p_1 p_2})/\mathbb{k}$, then Lemma 7 implies that

$$\begin{aligned} \left(\frac{q}{\mathcal{H}_1 \mathcal{H}_2} \right) &= \left(\frac{q}{\mathcal{H}_1} \right) \left(\frac{q}{\mathcal{H}_2} \right) \\ &= \left(\frac{q}{p_1} \right) \left(\frac{q}{p_1} \right) \\ &= 1. \end{aligned}$$

$$\begin{aligned} \left(\frac{q}{\mathcal{H}_1 \mathcal{H}_3} \right) &= \left(\frac{q}{\mathcal{H}_1} \right) \left(\frac{q}{\mathcal{H}_3} \right) \\ &= \left(\frac{q}{p_1} \right) \left(\frac{q}{p_2} \right) \\ &= 1. \end{aligned}$$

Consequently

$$N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3)) = \langle [\mathcal{H}_1 \mathcal{H}_2], [\mathcal{H}_1 \mathcal{H}_3] \rangle,$$

since

$$N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3)) = \{[\mathcal{H}] \in \mathbf{Cl}_2(\mathbb{k}) / \left(\frac{2}{[\mathcal{H}]} = 1 \right)\}.$$

Let us determine $\kappa_{\mathbb{K}_3/\mathbb{k}}$. We know, from Lemma 2, that $E_{\mathbb{K}_3} = \langle i, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_q \varepsilon_{p_1 p_2 q}}, \sqrt{i \varepsilon_q} \rangle$ and that $E_{\mathbb{k}} = \langle i, \varepsilon_{p_1 p_2 q} \rangle$, hence $N_{\mathbb{K}_3/\mathbb{k}}(E_{\mathbb{K}_3}) = \langle i, \varepsilon_{p_1 p_2 q} \rangle$, thus $[E_{\mathbb{k}} : N_{\mathbb{K}_3/\mathbb{k}}(E_{\mathbb{K}_3})] = 1$, and $\#\kappa_{\mathbb{K}_3/\mathbb{k}} = 2$.

(a) If $N(\varepsilon_{p_1 p_2}) = 1$, then putting $\varepsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$ and applying Lemma 1, we get $\sqrt{2\varepsilon_{p_1 p_2}} = b_1\sqrt{2p_1} + b_2\sqrt{2p_2}$, where $b = 2b_1 b_2$. This implies that $p_1 \varepsilon_{p_1 p_2}$ is a square in \mathbb{K}_3 , therefore there exist $\alpha \in \mathbb{K}_3$ such that $(p_1) = (\alpha^2)$. As $(\mathcal{H}_1 \mathcal{H}_2)^2 = (p_1)$, so $\mathcal{H}_1 \mathcal{H}_2 = (\alpha)$, which yields that $\mathcal{H}_1 \mathcal{H}_2$ capitulates in \mathbb{K}_3 .

(b) If $N(\varepsilon_{p_1 p_2}) = -1$ and since $p_1 p_2 \equiv 1 \pmod{8}$, then, by decomposition uniqueness in $\mathbb{Z}[i]$, there exist b_1 and b_2 in $\mathbb{Z}[i]$ such that:

$$\begin{cases} a \mp i = ib_1^2 \pi_1 \pi_3, \\ a \pm i = -ib_2^2 \pi_2 \pi_4, \end{cases} \quad \text{or} \quad \begin{cases} a \mp i = ib_1^2 \pi_2 \pi_3, \\ a \pm i = -ib_2^2 \pi_1 \pi_4, \end{cases} \quad \text{hence}$$

$$\sqrt{\varepsilon_{p_1 p_2}} = b_1 \sqrt{\pi_1 \pi_3} + b_2 \sqrt{\pi_2 \pi_4} \quad \text{or} \quad \sqrt{\varepsilon_{p_1 p_2}} = b_1 \sqrt{\pi_2 \pi_3} + b_2 \sqrt{\pi_1 \pi_4},$$

where $p_1 = \pi_1 \pi_2$, $p_2 = \pi_3 \pi_4$ and π_j are in $\mathbb{Z}[i]$. Thus

$$\sqrt{\pi_1 \pi_3 \varepsilon_{p_1 p_2}} = b_1 \pi_1 \pi_3 + b_2 \sqrt{p_1 p_2} \quad \text{or} \quad \sqrt{\pi_2 \pi_3 \varepsilon_{p_1 p_2}} = b_1 \pi_2 \pi_3 + b_2 \sqrt{p_1 p_2},$$

so there exist α, β in \mathbb{K}_3 such $(\pi_1 \pi_3) = (\alpha^2)$ or $(\pi_2 \pi_3) = (\beta^2)$. This yields that $\mathcal{H}_1 \mathcal{H}_3 = (\alpha)$ or $\mathcal{H}_2 \mathcal{H}_3 = (\beta)$ i.e. $\kappa_{\mathbb{K}_3/\mathbb{k}} = \langle [\mathcal{H}_1 \mathcal{H}_3] \rangle$ or $\kappa_{\mathbb{K}_3/\mathbb{k}} = \langle [\mathcal{H}_2 \mathcal{H}_3] \rangle$. On the other hand, $\mathcal{H}_1 \mathcal{H}_3$ is not principal in \mathbb{k} . In fact, there exist x and y in \mathbb{Z} such that $(\mathcal{H}_1 \mathcal{H}_3)^2 = (\pi_1 \pi_3) = (x + iy)$, hence $\sqrt{x^2 + y^2} = \sqrt{p_1 p_2} \notin \mathbb{k}$. Thus [7, Proposition 1] implies the result. Similarly, we show that $\mathcal{H}_1 \mathcal{H}_4$, $\mathcal{H}_2 \mathcal{H}_4$ and $\mathcal{H}_2 \mathcal{H}_3$ are not principal in \mathbb{k} .

As $p_1 p_2 (x \pm 1)$ is a square in \mathbb{N} (Lemma 1), so it is easy to see that $\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4$ is principal in \mathbb{k} , hence $\kappa_{\mathbb{K}_3/\mathbb{k}} = \langle [\mathcal{H}_1 \mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_2 \mathcal{H}_3] \rangle$. Consequently

$$\kappa_{\mathbb{K}_3/\mathbb{k}} = \begin{cases} \langle [\mathcal{H}_1 \mathcal{H}_2] \rangle = \langle [\mathcal{H}_3 \mathcal{H}_4] \rangle & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \langle [\mathcal{H}_1 \mathcal{H}_3] \rangle \text{ or } \langle [\mathcal{H}_2 \mathcal{H}_3] \rangle & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

From the Figure 3, we see that \mathbb{K}_3/F_1 and \mathbb{K}_3/F_2 are ramified extensions, whereas \mathbb{K}_3/\mathbb{k} is not. Therefore, by the class field theory, we deduce that $[\mathbf{Cl}_2(\mathbb{k}) : N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3))] = 2$, $\mathbf{Cl}_2(F) = N_{\mathbb{K}_3/F}(\mathbf{Cl}_2(\mathbb{K}_3))$ and $\mathbf{Cl}_2(F_1) = N_{\mathbb{K}_3/F_1}(\mathbf{Cl}_2(\mathbb{K}_3))$, thus Theorem 5 implies that

$$N_{\mathbb{K}_3/F_1}(\mathbf{Cl}_2(\mathbb{K}_3)) = \langle [2_{F_1}], [I] \rangle.$$

Hence there exist ideals \mathfrak{P} and \mathfrak{A} in \mathbb{K}_3 such that $N_{\mathbb{K}_3/F_1}(\mathfrak{P}) \sim I$, $N_{\mathbb{K}_3/F_1}(\mathfrak{A}) \sim 2_{F_1}$, $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \in \mathbf{Cl}_2(\mathbb{k})$ and $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \in \mathbf{Cl}_2(\mathbb{k})$. Note that \mathfrak{A} is an ideal in \mathbb{K}_3 above 2. We prove that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1 \mathcal{H}_2$ and $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1 \mathcal{H}_3$ or $\mathcal{H}_2 \mathcal{H}_3$ (see Lemma 8 below). Thus we claim that

$$\begin{cases} \mathfrak{P}^2 \sim I, & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \mathfrak{P}^2 \sim \mathcal{H}_1 \mathcal{H}_2 I, & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

and

$$\begin{cases} \mathfrak{A}^2 \sim \mathcal{H}_1 \mathcal{H}_3 2_{F_1}, & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \mathfrak{A}^2 \sim 2_{F_1}, & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

Before showing this, note that in $\mathbf{Cl}_2(F_1)$ and in the case where $N(\varepsilon_{p_1 p_2}) = 1$ we have: $\mathfrak{p}_1 \sim \mathcal{P}_1 \mathcal{P}_2 \sim 1$ and $\mathfrak{p}_2 \sim \mathcal{P}_3 \mathcal{P}_4 \sim 1$, hence $\mathcal{P}_1 \sim \mathcal{P}_2$ and $\mathcal{P}_3 \sim \mathcal{P}_4$. Therefore, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we get $\mathcal{H}_1 \sim \mathcal{H}_2$ and $\mathcal{H}_3 \sim \mathcal{H}_4$.

To this end, let t and s be the elements of order 2 in $\text{Gal}(\mathbb{K}_3/\mathbb{Q}(i))$ which fix F_1 and \mathbb{k} respectively. Using the identity $2 + (1+t+s+ts) = (1+t) + (1+s) + (1+ts)$ of the group ring $\mathbb{Z}[\text{Gal}(F_1/\mathbb{Q})]$ and observing that $\mathbb{Q}(i)$ and F_2 have odd class numbers we find:

$$\mathfrak{P}^2 \sim \mathfrak{P}^{1+t} \mathfrak{P}^{1+s} \mathfrak{P}^{1+ts} \sim \mathcal{H}_1 \mathcal{H}_2 I \text{ and}$$

$$\mathfrak{A}^2 \sim \mathfrak{A}^{1+t} \mathfrak{A}^{1+s} \mathfrak{A}^{1+ts} \sim \mathcal{H}_1 \mathcal{H}_3 2_{F_1} \text{ or } \mathcal{H}_2 \mathcal{H}_3 2_{F_1}.$$

As, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we have $\mathcal{H}_1 \mathcal{H}_2 \sim 1$ if $N(\varepsilon_{p_1 p_2}) = 1$ and $\mathcal{H}_1 \mathcal{H}_3 \sim 1$ or $\mathcal{H}_2 \mathcal{H}_3 \sim 1$ if $N(\varepsilon_{p_1 p_2}) = -1$, so the result claimed. Thus Theorem 5 implies that, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we have:

$$\begin{cases} \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim \mathfrak{p}_1 \sim 1 & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim \mathfrak{p}_1 \not\sim 1 \text{ and } \mathfrak{P}^{2^{m+1}} \sim 1 & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

and

$$\begin{cases} \mathfrak{A}^{2^{n+2}} \sim 2_{F_1}^{2^{n+1}} \sim 1 & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \mathfrak{A}^{2^{n+1}} \sim 2_{F_1}^{2^n} \not\sim 1 \text{ and } \mathfrak{A}^{2^{n+2}} \sim 1 & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

Moreover, Lemma 4 and Theorem 5 yield that:

- If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then

$$\begin{cases} \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim 2_{F_1}^{2^n} \sim \mathfrak{A}^{2^{n+1}} \sim \mathfrak{p}_1 \sim \mathcal{H}_1 \mathcal{H}_2, \text{ and} \\ \mathcal{H}_1 \sim \mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}} \sim \mathfrak{A}^{2^n} \mathfrak{P}^{2^{m-1}}. \end{cases}$$

- If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then

$$\begin{cases} \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim 2_{F_1}^{2^{n+1}} \sim \mathfrak{A}^{2^{n+2}} \sim \mathfrak{p}_1 \sim \mathcal{H}_1 \mathcal{H}_2 \sim 1, \\ \mathcal{H}_1 \mathcal{H}_3 \sim \mathcal{P}_1 \mathcal{P}_3 \sim 2_{F_1}^{2^n} \sim \mathfrak{A}^{2^{n+1}}, \\ \mathcal{H}_1 \sim \mathfrak{A}^{2^{n+1}} \mathfrak{P}^{2^{m-1}} \text{ and } \mathcal{H}_3 \sim \mathfrak{P}^{2^{m-1}} \text{ or } \mathcal{H}_3 \sim \mathfrak{A}^{2^{n+1}} \mathfrak{P}^{2^{m-1}} \text{ and } \mathcal{H}_1 \sim \mathfrak{P}^{2^{m-1}}. \end{cases}$$

Finally, note that for all $i \leq n$ and $j \leq m-1$, we have $\mathfrak{A}^{2^i} \mathfrak{P}^{2^j} \not\sim 1$. As otherwise we would have, by applying the norm $N_{\mathbb{K}_3/F_1}$, $2_{F_1}^{2^i} \mathfrak{P}^{2^{j+1}} \sim 1$, which is absurd.

Taking into account Lemma 5, and since the 2-rank of \mathbb{K}_3 is 2 and its 2-class number is $h(\mathbb{K}_3) = 2h(p_1 p_2)h(-p_1 p_2)$ (see Lemmas 2 and 6), so the results that we have just prove, we get the following conclusion:

Conclusion

• If $\left(\frac{p_1}{p_2}\right) = -1$, then $\langle [\mathfrak{A}], [\mathfrak{P}] \rangle$ is a subgroup of $\mathbf{Cl}_2(\mathbb{K}_3)$ of type $(2^{m+1}, 2^{n+1})$, thus

$$\mathbf{Cl}_2(\mathbb{K}_3) = \langle [\mathfrak{A}], [\mathfrak{P}] \rangle \simeq (2^{n+1}, 2^{m+1}) = (2^2, 2^{m+1}).$$

• If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $\langle [\mathfrak{A}], [\mathfrak{P}] \rangle$ is a subgroup of $\mathbf{Cl}_2(\mathbb{K}_3)$ of type $(2^{\min(m, n+1)}, 2^{\max(m+1, n+2)}) = (2^m, 2^{n+2})$, thus

$$\mathbf{Cl}_2(\mathbb{K}_3) = \langle [\mathfrak{A}], [\mathfrak{P}] \rangle \sim (2^m, 2^{n+2}) = (2^2, 2^{n+2}).$$

• If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\langle [\mathfrak{A}], [\mathfrak{P}] \rangle$ is a subgroup of $\mathbf{Cl}_2(\mathbb{K}_3)$ of type $(2^m, 2^{n+2})$, thus

$$\mathbf{Cl}_2(\mathbb{K}_3) = \langle [\mathfrak{A}], [\mathfrak{P}] \rangle \simeq (2^m, 2^{n+2}).$$

Moreover

$$\mathbf{Cl}_2(\mathbb{K}_3) \simeq \begin{cases} (2^m, 2^3) & \text{if } \left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1, \\ (2^2, 2^{n+2}) & \text{if } \left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1 \end{cases}$$

(4) **Computation of $\text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$.** Put $L = \mathbb{k}_2^{(2)}$, the Hilbert 2-class field of \mathbb{k} , and denote by $\left(\frac{L/\mathbb{K}_3}{P}\right)$ the Artin symbol for the normal extension L/\mathbb{K}_3 , then $\sigma = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}}\right)$ and $\tau = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}}\right)$ generate the abelian subgroup $\text{Gal}(L/\mathbb{K}_3)$ of $G = \text{Gal}(L/\mathbb{k})$. Put also $\rho = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1}\right)$, then ρ restricts to the nontrivial

automorphism of \mathbb{K}_3/\mathbb{k} , since \mathcal{H}_1 is not norm in \mathbb{K}_3/\mathbb{k} (\mathcal{H}_1 remains inert in \mathbb{K}_3). Thus

$$G = \text{Gal}(L/\mathbb{k}) = \langle \rho, \tau, \sigma \rangle.$$

For the Artin symbol properties we can see [16].

Note finally that $|G| = 2|\text{Gal}(L/\mathbb{K}_3)| = 2^{n+m+3}$.

To continue we need the following result:

Lemma 8. *In $\text{Cl}_2(\mathbb{k})$, we have $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2$ and $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$ or $\mathcal{H}_2\mathcal{H}_3$. Moreover*

- (1) Assume $\left(\frac{p_1}{p_2}\right) = -1$ and put $I = \left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4$.
 - (i) If $I = \left(\frac{\pi_1}{\pi_3}\right)$, then $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$.
 - (ii) If $I = -\left(\frac{\pi_1}{\pi_3}\right)$, then $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_2\mathcal{H}_3$.
- (2) Assume $\left(\frac{p_1}{p_2}\right) = 1$ and put $\beta = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right)$.
 - (i) If $\beta = 1$, then $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$.
 - (ii) If $\beta = -1$, then $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_2\mathcal{H}_3$.

Proof. Recall that $N_{\mathbb{K}_3/\mathbb{k}}(\text{Cl}_2(\mathbb{K}_3)) = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$. Choose a prime ideal \mathfrak{A} in \mathbb{K}_3 such that $[\mathfrak{A}] = [\mathfrak{P}]$, this is always possible by Chebotarev's theorem, thus $\mathcal{R} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})$ is a prime ideal in \mathbb{k} . If $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_3$, then $\mathcal{R} \sim \mathcal{H}_1\mathcal{H}_3$ (equivalence in $\text{Cl}_2(\mathbb{k})$). Hence the prime ideal $\mathfrak{r} = N_{\mathbb{k}/k_0}(\mathcal{R})$ of k_0 is equivalent, in $\text{Cl}_2(k_0)$, to $\tilde{\mathfrak{z}} \sim P_1 P_2 \sim \tilde{\mathfrak{q}}$. It should be noted that $\text{Cl}_2(k_0)$ is of type (2, 2) and it is generated by P_1 and P_2 , the prime ideals of k_0 above p_1 and p_2 respectively. Therefore $\mathfrak{r} \sim \tilde{\mathfrak{q}}$ and $\mathfrak{r} \sim P_1 P_2$, these imply that $\pm r q = X^2 - y^2 p_1 p_2 q$ and $\pm r p_1 p_2 = U^2 - v^2 p_1 p_2 q$, with some X, y, U and v in \mathbb{Z} . Putting $X = xq$ and $U = up_1 p_2$, we get $\pm r = x^2 q - y^2 p_1 p_2$ and $\pm r = u^2 p_1 p_2 - v^2 q$, from which we deduce that $\left(\frac{\pm r}{q}\right) = \left(\frac{-p_1 p_2}{q}\right) = -1$ and $\left(\frac{\pm r}{q}\right) = 1$, which is a contradiction. Similarly, we show that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \not\sim \mathcal{H}_2\mathcal{H}_3$. Finally, the equivalence $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim 1$ can not occur since the order of σ is strictly greater than 1. Consequently $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2$.

Let \mathcal{H}_0 denote a prime ideal of \mathbb{k} above 2. If $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_2$, then $\mathcal{H}_0 \sim \mathcal{H}_1\mathcal{H}_2$, thus $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2 \sim 1$. Hence $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2 = (\alpha)$, with some $\alpha \in \mathbb{k}$. We have two cases to distinguish:

1st case: If $q \equiv 3 \pmod{8}$, then there is only one prime ideal \mathcal{H}_0 in \mathbb{k} above 2, thus $(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2)^2 = (2p_1)$. Therefore $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$ is principal in \mathbb{k} if and only if $p_1(x \pm 1)$ or $2p_1(x \pm 1)$ is a square in \mathbb{N} (see [7, Remark 1]), which is not the case (Lemma 1).

2nd case: If $q \equiv 7 \pmod{8}$, then there are two prime ideals \mathcal{H}_0 and \mathcal{H}'_0 in \mathbb{k} above 2. Thus $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2 = (\alpha)$ implies, by applying the norm $N_{\mathbb{k}/k_0}$, that $P_0 = (\alpha')$, where P_0 is the prime ideal of k_0 above 2, hence $(2) = (\alpha'^2)$, this in turn yields that $2\varepsilon_{p_1p_2q}$ is a square in $\mathbb{Q}(\sqrt{p_1p_2q})$ i.e. $x \pm 1$ is a square in \mathbb{N} , which is not the case (Lemma 1). Finally, the equivalence $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim 1$ can not occur since the order of τ is strictly greater than 1. From which we conclude that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$ or $\mathcal{H}_2\mathcal{H}_3$.

Suppose that $I = -1$ and $\left(\frac{\pi_1}{\pi_3}\right) = 1$. If $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$, then $N_{\mathbb{k}/\mathbb{Q}(i)}(N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})) \sim \pi_1\pi_3$. On the other hand, \mathfrak{A} is a prime ideal of \mathbb{K}_3 above 2, thus $N_{\mathbb{k}/\mathbb{Q}(i)}(N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})) \sim 1+i$. Hence $\pi_1\pi_3 \sim 1+i$. Therefore Hilbert symbol properties and Lemma 7 imply that $\left(\frac{\pi_2\pi_4}{\mathcal{H}_1\mathcal{H}_3}\right) = \left(\frac{\pi_2\pi_4}{\pi_1\pi_3}\right) = \left(\frac{\pi_2\pi_4}{1+i}\right) = \left(\frac{1+i}{\pi_2}\right)\left(\frac{1+i}{\pi_4}\right)$. Thus Lemma 9 (see below) yields that $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1$. Which is absurd, since, according to [8, Proposition 1], $I = \left(\frac{\pi_1}{\pi_3}\right)\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right)$. The other assertions are proved similarly using \square .

We can now establish the following relations (equivalences are in $\mathbf{Cl}_2(\mathbb{K}_3)$):

- $[\tau, \sigma] = 1$.
 - $\rho^2 = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1^2}\right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathcal{H}_1)}\right) = \left(\frac{L/\mathbb{K}_3}{\mathcal{H}_1}\right)$, thus $\rho^4 = 1$.
 - $\tau\rho^{-1}\tau\rho = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}^{1+\rho}}\right) = \begin{cases} 1 & \text{if } N(\varepsilon_{p_1p_2}) = -1, \\ \tau^{2^{n+1}} & \text{if } N(\varepsilon_{p_1p_2}) = 1, \end{cases}$
- because $\mathfrak{A}^{1+\rho} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3 \sim \mathfrak{A}^{2^{n+1}}$ if $N(\varepsilon_{p_1p_2}) = 1$, otherwise we get $\mathfrak{A}^{1+\rho} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$ or $\mathcal{H}_2\mathcal{H}_3$. Since $\mathcal{H}_1\mathcal{H}_3$ and $\mathcal{H}_2\mathcal{H}_3$ are norms in \mathbb{K}_3/\mathbb{k} , so, with out loss of generality, we can assume that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim 1$, because $\kappa_{\mathbb{K}_3/\mathbb{k}} = \langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$. Thus
- $[\tau, \rho] = \begin{cases} \tau^{-1}\rho^{-1}\tau\rho = \tau^{-2} & \text{if } N(\varepsilon_{p_1p_2}) = -1, \\ \tau^{2^{n+1}-2} & \text{if } N(\varepsilon_{p_1p_2}) = 1. \end{cases}$
- $\sigma\rho^{-1}\sigma\rho = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}^{1+\rho}}\right) = \begin{cases} 1 & \text{if } N(\varepsilon_{p_1p_2}) = 1, \\ \sigma^{2^m} & \text{if } N(\varepsilon_{p_1p_2}) = -1; \end{cases}$ since $\mathfrak{P}^{1+\rho} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2 \sim \begin{cases} 1 & \text{if } N(\varepsilon_{p_1p_2}) = 1, \\ \mathfrak{P}^{2^m} & \text{if } N(\varepsilon_{p_1p_2}) = -1. \end{cases}$ Thus
 - $[\sigma, \rho] = \begin{cases} \sigma^{-2} & \text{if } N(\varepsilon_{p_1p_2}) = 1, \\ \sigma^{2^m-2} & \text{if } N(\varepsilon_{p_1p_2}) = -1. \end{cases}$
 - If $\left(\frac{p_1}{p_2}\right) = -1$, then $n = 1$, $m \geq 2$ and

$$\begin{cases} \rho^4 = \sigma^{2^{m+1}} = \tau^{2^{n+2}} = \tau^{2^3} = 1, \\ \sigma^{2^m} = \tau^{2^{n+1}} = \tau^4, \\ \rho^2 = \tau^{2^n}\sigma^{2^{m-1}} = \tau^2\sigma^{2^{m-1}}; \end{cases} \quad \text{since } \mathfrak{A}^{2^{n+2}} \sim \mathfrak{P}^{2^{m+1}} \sim 1, \mathcal{H}_1 \sim \mathfrak{A}^{2^n}\mathfrak{P}^{2^{m-1}}$$

and $\mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m}$. Moreover $[\tau, \rho] = \tau^{-2}$ and $[\sigma, \rho] = \sigma^{2^m-2}$.

- If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $m = 2$, $n \geq 2$ and

$$\begin{cases} \rho^4 = \tau^{2^{n+2}} = \sigma^{2^{m+1}} = \sigma^{2^3} = 1, \\ \tau^{2^{n+1}} = \sigma^{2^m} = \sigma^4, \\ \rho^2 = \tau^{2^n} \sigma^{2^{m-1}} = \tau^{2^n} \sigma^2; \end{cases} \quad \text{since } \mathfrak{A}^{2^{n+2}} \sim \mathfrak{P}^{2^{m+1}} \sim 1, \mathcal{H}_1 \sim \mathfrak{A}^{2^n} \mathfrak{P}^{2^{m-1}}$$

and $\mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m}$. Moreover $[\tau, \rho] = \tau^{-2}$ and $[\sigma, \rho] = \sigma^2$.

- If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $m \geq 2$, $n \geq 1$ and

$$\begin{cases} \rho^4 = \tau^{2^{n+2}} = \sigma^{2^m} = 1, \\ \rho^2 = \tau^{2^{n+1}} \sigma^{2^{m-1}} \text{ or } \rho^2 = \sigma^{2^{m-1}}; \end{cases} \quad \text{since } \mathfrak{A}^{2^{n+2}} \sim \mathfrak{P}^{2^m} \sim 1 \text{ and } \mathcal{H}_1 \sim \mathfrak{A}^{2^{n+1}} \mathfrak{P}^{2^{m-1}}$$

or $\mathcal{H}_1 \sim \mathfrak{P}^{2^{m-1}}$. Moreover $[\tau, \rho] = \tau^{2^{n+1}-2}$ and $[\sigma, \rho] = \sigma^{-2}$.

(5) **Types of $\text{Cl}_2(\mathbb{k}_2^{(1)})$.** We know that $[\tau, \sigma] = 1$, $[\sigma, \rho] = \sigma^{-2}$ or σ^{2^m-2} and $[\tau, \rho] = \tau^{-2}$ or $\tau^{2^{n+1}-2}$, and since $\langle \sigma^{2^m-2} \rangle \simeq \langle \sigma^{-2} \rangle$ and $\langle \tau^{2^{n+1}-2} \rangle \simeq \langle \tau^{-2} \rangle$, then $G' \simeq \langle \sigma^2, \tau^2 \rangle$, where G' is the derived group of G , hence

$$\text{Cl}_2(\mathbb{k}_2^{(1)}) \simeq \begin{cases} (2, 2^m) & \text{if } \left(\frac{p_1}{p_2}\right) = -1, \\ (2, 2^{n+1}) & \text{if } \left(\frac{p_1}{p_2}\right) = 1 \text{ and } N(\varepsilon_{p_1 p_2}) = -1, \\ (2^{m-1}, 2^2) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, N(\varepsilon_{p_1 p_2}) = 1 \text{ and } \left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1, \\ (2, 2^{n+1}) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, N(\varepsilon_{p_1 p_2}) = 1 \text{ and } \left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1. \end{cases}$$

(6) **The coclass of G .** The lower central series of G is defined inductively by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$, that is the subgroup of G generated by the set $\{[a, b] = a^{-1}b^{-1}ab/a \in \gamma_i(G), b \in G\}$, so the coclass of G is defined to be $cc(G) = h - c$, where $|G| = 2^h$ and $c = c(G)$ is the nilpotency class of G , that is the smallest positive integer c satisfying $\gamma_{c+1}(G) = 1$. We easily get

$$\gamma_1(G) = G.$$

$$\gamma_2(G) = G' = \langle \sigma^2, \tau^2 \rangle.$$

$$\gamma_3(G) = [G', G] = \langle \sigma^4, \tau^4 \rangle.$$

Then Proposition 1(6) (below) implies that $\gamma_{j+1}(G) = [\gamma_j(G), G] = \langle \sigma^{2^j}, \tau^{2^j} \rangle$.

If $\left(\frac{p_1}{p_2}\right) = -1$, then $\gamma_{m+2}(G) = \langle \sigma^{2^{m+1}}, \tau^{2^{m+1}} \rangle = \langle 1 \rangle$ and $\gamma_{m+1}(G) = \langle \sigma^{2^m}, \tau^{2^m} \rangle \neq \langle 1 \rangle$. Since $|G| = 2^{n+m+3}$, we have

$$c(G) = m + 1 \text{ and } cc(G) = n + m + 3 - m - 1 = 3.$$

If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $\gamma_{n+3}(G) = \langle \sigma^{2^{n+2}}, \tau^{2^{n+2}} \rangle = \langle 1 \rangle$ and $\gamma_{n+2}(G) = \langle \sigma^{2^{n+1}}, \tau^{2^{n+1}} \rangle \neq \langle 1 \rangle$. As $|G| = 2^{n+m+3}$, so

$$c(G) = n + 2 \text{ and } cc(G) = n + m + 3 - n - 2 = 3.$$

If $\left(\frac{p_1}{p_2}\right) = 1$, $N(\varepsilon_{p_1 p_2}) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then $n = 1$, $m \geq 3$, $\gamma_{m+1}(G) = \langle \sigma^{2^m}, \tau^{2^m} \rangle = \langle 1 \rangle$ and $\gamma_m(G) = \langle \sigma^{2^{m-1}}, \tau^{2^{m-1}} \rangle \neq \langle 1 \rangle$. As $|G| = 2^{n+m+3}$, so

$$c(G) = m \text{ and } cc(G) = n + m + 3 - m = 4.$$

If $\left(\frac{p_1}{p_2}\right) = 1$, $N(\varepsilon_{p_1 p_2}) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$, then $m = 2$, $n \geq 2$, $\gamma_{n+3}(G) = \langle \sigma^{2^{n+2}}, \tau^{2^{n+2}} \rangle = \langle 1 \rangle$ and $\gamma_{n+2}(G) = \langle \sigma^{2^{n+1}}, \tau^{2^{n+1}} \rangle \neq \langle 1 \rangle$. As $|G| = 2^{n+m+3}$, so

$$c(G) = n + 2 \text{ and } cc(G) = n + m + 3 - n - 2 = 3.$$

5.2. Proof of Theorems 3 and 4. For this we need the following result.

Proposition 1. *Let $G = \langle \sigma, \tau, \rho \rangle$ be the group defined above.*

- (1) $\rho^{-1} \tau \rho = \begin{cases} \tau^{-1} & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ \tau^{2^{n+1}-1} & \text{if } N(\varepsilon_{p_1 p_2}) = 1. \end{cases}$
- (2) $\rho^{-1} \sigma \rho = \sigma^{-1}$.
- (3) $[\rho^2, \sigma] = [\rho^2, \tau] = 1$.
- (4) $(\sigma \tau \rho)^2 = (\tau \rho)^2 = \begin{cases} \rho^2 & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ \rho^2 \tau^{2^{n+1}} & \text{if } N(\varepsilon_{p_1 p_2}) = 1. \end{cases}$
- (5) $(\sigma \rho)^2 = \rho^2$.
- (6) For all $r \in \mathbb{N}^*$, we have $[\rho, \tau^{2^r}] = \tau^{2^{r+1}}$ and $[\rho, \sigma^{2^r}] = \sigma^{2^{r+1}}$.

Proof. (6) Since $[\rho, \tau] = \begin{cases} \tau^2 & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ \tau^{2-2^{n+1}} & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \end{cases}$

$$\text{so } [\rho, \tau^2] = \begin{cases} \tau^4 & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ \tau^{2-2^{n+1}} \tau^{2-2^{n+1}} = \tau^4 & \text{if } N(\varepsilon_{p_1 p_2}) = 1. \end{cases}$$

By induction, we show that for all $r \in \mathbb{N}^*$, $[\rho, \tau^{2^r}] = \tau^{2^{r+1}}$. Similarly, we get that $[\rho, \sigma^{2^r}] = \sigma^{2^{r+1}}$. \square

The proof of Theorems 3 and 4 consists of 3 parts. In the first part, we will compute $N_{\mathbb{K}_j/\mathbb{K}}(\mathbf{Cl}_2(\mathbb{K}_j))$, for all $1 \leq j \leq 7$. In the second one, we will determine the capitulation kernels $\kappa_{\mathbb{K}_j}$ and the types of $\mathbf{Cl}_2(\mathbb{K}_j)$ and in the third one, we will determine the capitulation kernels $\kappa_{\mathbb{L}_j}$ and the types of $\mathbf{Cl}_2(\mathbb{L}_j)$.

5.2.1. Norm class groups. Let us compute $N_j = N_{\mathbb{K}_j/\mathbb{K}}(\mathbf{Cl}_2(\mathbb{K}_j))$, the results are summarized in the following table.

Table 1: Norm class groups

\mathbb{K}_j	Conditions	N_j for $\left(\frac{p_1}{p_2}\right) = 1$	N_j for $\left(\frac{p_1}{p_2}\right) = -1$
\mathbb{K}_1		$\langle [\mathcal{H}_3], [\mathcal{H}_1 \mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$
\mathbb{K}_2		$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1 \mathcal{H}_2], [\mathcal{H}_3] \rangle$
\mathbb{K}_3		$\langle [\mathcal{H}_1 \mathcal{H}_3], [\mathcal{H}_2 \mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1 \mathcal{H}_3], [\mathcal{H}_2 \mathcal{H}_3] \rangle$
\mathbb{K}_4	$\pi = 1$	$\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_1 \mathcal{H}_3] \rangle$
	$\pi = -1$	$\langle [\mathcal{H}_2], [\mathcal{H}_1 \mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$

\mathbb{K}_j	Conditions	N_j for $\left(\frac{p_1}{p_2}\right) = 1$	N_j for $\left(\frac{p_1}{p_2}\right) = -1$
\mathbb{K}_5	$\pi = 1$ $\pi = -1$	$\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$
\mathbb{K}_6	$\pi = 1$ $\pi = -1$	$\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$
\mathbb{K}_7	$\pi = 1$ $\pi = -1$	$\langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$ $\langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$

To check the table entries we use Lemma 7 and the following results which are easy to prove.

Lemma 9. *Let $p_1 \equiv p_2 \equiv 1 \pmod{4}$ be different primes. Put $p_1 = \pi_1\pi_2$ and $p_2 = \pi_3\pi_4$, where $\pi_j \in \mathbb{Z}[i]$, then*

- (i) $\left(\frac{\pi_1}{\pi_2}\right) = \left(\frac{\pi_3}{\pi_4}\right) = \begin{cases} 1 & \text{if } p_1 \equiv p_2 \equiv 1 \pmod{8}, \\ -1 & \text{if } p_1 \equiv p_2 \equiv 5 \pmod{8} \end{cases}$
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $\left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{\pi_2}{\pi_3}\right) = \left(\frac{\pi_1}{\pi_4}\right) = \left(\frac{\pi_2}{\pi_4}\right)$.
- (iii) If $\left(\frac{p_1}{p_2}\right) = -1$, then $\left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{\pi_2}{\pi_4}\right) = -\left(\frac{\pi_2}{\pi_3}\right) = -\left(\frac{\pi_1}{\pi_4}\right)$.
- (iv) If $\left(\frac{2}{p_1}\right) = 1$, then $\left(\frac{1+i}{\pi_1}\right) = \left(\frac{1+i}{\pi_2}\right)$.
- (v) If $\left(\frac{2}{p_1}\right) = -1$, then $\left(\frac{1+i}{\pi_1}\right) = -\left(\frac{1+i}{\pi_2}\right)$.

Compute N_j in a few cases keeping in mind that \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 are unramified prime ideals in \mathbb{K}_j/\mathbb{k} .

- Take as a first example: $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1}) = \mathbb{k}(\sqrt{p_2q}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2q}, i)$. As $N_1 = \{[\mathcal{H}] \in \mathbf{Cl}_2(\mathbb{k}) / \left(\frac{\alpha}{\mathcal{H}}\right) = 1\}$, so for all $j \in \{1, 2\}$ we get

$$\begin{aligned}
\left(\frac{\mathbb{k}(\sqrt{p_2q})/\mathbb{k}}{\mathcal{H}_j}\right) &= \left(\frac{\mathbb{k}(\sqrt{p_2q})/\mathbb{k}}{\mathcal{H}_j}\right) (\sqrt{p_2q})(\sqrt{p_2q})^{-1} \\
&= \left(\frac{p_2q}{\mathcal{H}_j}\right) \\
&= \left(\frac{p_2q}{p_1}\right) \\
&= \left(\frac{q}{p_1}\right) \left(\frac{p_1}{p_2}\right) \\
&= -\left(\frac{p_1}{p_2}\right).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned} \left(\frac{\mathbb{k}(\sqrt{p_1})/\mathbb{k}}{\mathcal{H}_3} \right) &= \left(\frac{\mathbb{k}(\sqrt{p_1})/\mathbb{k}}{\mathcal{H}_3} \right) (\sqrt{p_1})(\sqrt{p_1})^{-1} \\ &= \left(\frac{p_1}{\mathcal{H}_3} \right) \\ &= \left(\frac{p_1}{p_2} \right). \text{ Thus} \end{aligned}$$

- If $\left(\frac{p_1}{p_2} \right) = -1$, then $[\mathcal{H}_j] \in N_1$ and $N_1 = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.
- If $\left(\frac{p_1}{p_2} \right) = 1$, then $[\mathcal{H}_j] \notin N_1$, $[\mathcal{H}_1\mathcal{H}_2] \in N_1$ and $[\mathcal{H}_3] \in N_1$. Hence $N_1 = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3] \rangle$.

• Take as a second example: $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3}) = \mathbb{k}(\sqrt{q\pi_2\pi_4})$.

1st case: Assume first that $\left(\frac{p_1}{p_2} \right) = -1$, hence Lemmas 7 and 9 imply that:

$$\begin{cases} \left(\frac{\pi_1\pi_3}{\mathcal{H}_2} \right) = \left(\frac{\pi_1\pi_3}{\pi_2} \right) = \left(\frac{\pi_1}{\pi_2} \right) \left(\frac{\pi_3}{\pi_2} \right) = - \left(\frac{\pi_2}{\pi_3} \right) = \left(\frac{\pi_1}{\pi_3} \right), \\ \left(\frac{\pi_2\pi_4q}{\mathcal{H}_1} \right) = \left(\frac{\pi_2\pi_4q}{\pi_1} \right) = \left(\frac{q}{p_1} \right) \left(\frac{\pi_1}{\pi_2} \right) \left(\frac{\pi_4}{\pi_1} \right) = \left(\frac{\pi_1}{\pi_4} \right) = - \left(\frac{\pi_1}{\pi_3} \right), \\ \left(\frac{\pi_2\pi_4q}{\mathcal{H}_3} \right) = \left(\frac{\pi_2\pi_4q}{\pi_3} \right) = \left(\frac{q}{p_2} \right) \left(\frac{\pi_2}{\pi_3} \right) \left(\frac{\pi_4}{\pi_3} \right) = \left(\frac{\pi_2}{\pi_3} \right) = - \left(\frac{\pi_1}{\pi_3} \right). \end{cases}$$

Thus

- If $\left(\frac{\pi_1}{\pi_3} \right) = -1$, then $\mathcal{H}_1 \in N_4$ and $\mathcal{H}_3 \in N_4$. Hence $N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$.
- If $\left(\frac{\pi_1}{\pi_3} \right) = 1$, then $\mathcal{H}_2 \in N_4$ and $\mathcal{H}_1\mathcal{H}_3 \in N_4$. Hence $N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$.

2nd case: Assume $\left(\frac{p_1}{p_2} \right) = 1$, then Lemmas 7 and 9 imply that:

$$\begin{cases} \left(\frac{\pi_1\pi_3}{\mathcal{H}_2} \right) = \left(\frac{\pi_1\pi_3}{\pi_2} \right) = \left(\frac{\pi_1}{\pi_2} \right) \left(\frac{\pi_3}{\pi_2} \right) = - \left(\frac{\pi_2}{\pi_3} \right) = - \left(\frac{\pi_1}{\pi_3} \right), \\ \left(\frac{\pi_2\pi_4q}{\mathcal{H}_1} \right) = \left(\frac{\pi_2\pi_4q}{\pi_1} \right) = \left(\frac{q}{p_1} \right) \left(\frac{\pi_1}{\pi_2} \right) \left(\frac{\pi_4}{\pi_1} \right) = \left(\frac{\pi_1}{\pi_4} \right) = \left(\frac{\pi_1}{\pi_3} \right), \\ \left(\frac{\pi_2\pi_4q}{\mathcal{H}_3} \right) = \left(\frac{\pi_2\pi_4q}{\pi_3} \right) = \left(\frac{q}{p_2} \right) \left(\frac{\pi_2}{\pi_3} \right) \left(\frac{\pi_4}{\pi_3} \right) = \left(\frac{\pi_2}{\pi_3} \right) = \left(\frac{\pi_1}{\pi_3} \right). \end{cases}$$

Thus

- If $\left(\frac{\pi_1}{\pi_3} \right) = 1$, then $\mathcal{H}_1 \in N_4$ and $\mathcal{H}_3 \in N_4$. Hence $N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$.
- If $\left(\frac{\pi_1}{\pi_3} \right) = -1$, then $\mathcal{H}_2 \in N_4$ and $\mathcal{H}_1\mathcal{H}_3 \in N_4$. Hence $N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$.

Proceeding similarly, we check the other table inputs.

5.2.2. Capitulation kernels $\kappa_{\mathbb{K}_j/\mathbb{k}}$ and $\mathbf{Cl}_2(\mathbb{K}_j)$. Let us compute the Galois groups $G_j = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{K}_j)$, the capitulation kernels $\kappa_{\mathbb{K}_j}$, $\kappa_{\mathbb{K}_j} \cap N_j$ and the types of $\mathbf{Cl}_2(\mathbb{K}_j)$. The results are summarized in the following Tables 2 and 3. Put $\beta = \left(\frac{1+i}{\pi_1} \right) \left(\frac{1+i}{\pi_3} \right)$, $\pi = \left(\frac{\pi_1}{\pi_3} \right)$, $N = N(\varepsilon_{p_1p_2})$ and $\mathbf{I} = \left(\frac{p_1p_2}{2} \right)_4 \left(\frac{2p_1}{p_2} \right)_4 \left(\frac{2p_2}{p_1} \right)_4$. Note that, in the Table 2 and for the column G_j , the left hand side (if it exists) refers to the case $\beta = 1$, while the right one refers to the case $\beta = -1$. Whereas, in the

Table 3 and for the same column, the left hand side (if it exists) refers to the case $I = 1$, while the right one refers to the case $I = -1$.

Table 2: $\kappa_{\mathbb{K}_j/\mathbb{k}}$ for the case $\left(\frac{p_1}{p_2}\right) = 1$.

\mathbb{K}_j	G_j	$\kappa_{\mathbb{K}_j/\mathbb{k}}$	$\kappa_{\mathbb{K}_j/\mathbb{k}} \cap N_j$	$Cl_2(\mathbb{K}_j)$
\mathbb{K}_1	$\langle \sigma, \tau\rho, \tau^2 \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2] \rangle$	$(2, 2, 2)$
\mathbb{K}_2	$\langle \sigma, \rho, \tau^2 \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2] \rangle$	$(2, 2, 2)$
\mathbb{K}_3 $N = 1$ $N = -1$	$\langle \tau, \sigma \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$\kappa_{\mathbb{K}_3/\mathbb{k}}$	$(2^m, 2^{n+2})$
\mathbb{K}_4 $\pi = 1$ $\pi = -1$	$\langle \tau, \rho \rangle$ $\langle \sigma\tau, \rho \rangle$ $\langle \tau, \sigma\rho, \sigma^2 \rangle$ $\langle \sigma\tau, \sigma\rho, \sigma^2 \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$	$N_4 = \kappa_{\mathbb{K}_4/\mathbb{k}}$ $\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$
\mathbb{K}_5 $\pi = 1$ $\pi = -1$	$\langle \sigma\tau, \rho \rangle$ $\langle \tau, \rho \rangle$ $\langle \sigma\rho, \sigma\tau, \sigma^2 \rangle$ $\langle \sigma\rho, \tau, \sigma^2 \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$	$N_5 = \kappa_{\mathbb{K}_5/\mathbb{k}}$ $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$
\mathbb{K}_6 $\pi = 1$ $\pi = -1$	$\langle \sigma\tau, \sigma\rho \rangle$ $\langle \tau, \sigma\rho \rangle$ $\langle \rho, \sigma\tau, \sigma^2 \rangle$ $\langle \rho, \tau, \sigma^2 \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$	$N_6 = \kappa_{\mathbb{K}_6/\mathbb{k}}$ $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$
\mathbb{K}_7 $\pi = 1$ $\pi = -1$	$\langle \tau, \sigma\rho \rangle$ $\langle \sigma\tau, \sigma\rho \rangle$ $\langle \rho, \tau, \sigma^2 \rangle$ $\langle \rho, \sigma\tau, \sigma^2 \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$	$N_7 = \kappa_{\mathbb{K}_7/\mathbb{k}}$ $\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$

Table 3: $\kappa_{\mathbb{K}_j/\mathbb{k}}$ for the case $\left(\frac{p_1}{p_2}\right) = -1$.

\mathbb{K}_j	G_j	$\kappa_{\mathbb{K}_j/\mathbb{k}}$	$\kappa_{\mathbb{K}_j/\mathbb{k}} \cap N_j$	$Cl_2(\mathbb{K}_j)$
\mathbb{K}_1	$\langle \sigma, \rho \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	$N_1 = \kappa_{\mathbb{K}_1/\mathbb{k}}$	$(2, 4)$
\mathbb{K}_2	$\langle \sigma, \tau\rho \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3] \rangle$	$N_2 = \kappa_{\mathbb{K}_2/\mathbb{k}}$	$(2, 4)$
\mathbb{K}_3	$\langle \sigma, \tau \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$\kappa_{\mathbb{K}_3/\mathbb{k}}$	$(4, 2^{m+1})$
\mathbb{K}_4 $\pi = 1$ $\pi = -1$	$\langle \tau, \sigma\rho, \sigma^2 \rangle$ $\langle \tau\rho, \sigma\tau, \tau^2 \rangle$ $\langle \sigma\tau, \rho \rangle$ $\langle \tau, \rho \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ $N_4 = \kappa_{\mathbb{K}_4/\mathbb{k}}$	$(2, 2, 2)$ $(2, 4)$
\mathbb{K}_5 $\pi = 1$ $\pi = -1$	$\langle \sigma\tau, \rho \rangle$ $\langle \tau, \rho \rangle$ $\langle \tau, \sigma\rho, \sigma^2 \rangle$ $\langle \sigma\tau, \sigma\rho, \tau^2 \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2\mathcal{H}_3] \rangle$	$N_5 = \kappa_{\mathbb{K}_5/\mathbb{k}}$ $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$
\mathbb{K}_6 $\pi = 1$ $\pi = -1$	$\langle \sigma\tau, \sigma\rho \rangle$ $\langle \sigma\rho, \tau \rangle$ $\langle \tau, \rho, \sigma^2 \rangle$ $\langle \rho, \sigma\tau, \tau^2 \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_3] \rangle$	$N_6 = \kappa_{\mathbb{K}_6/\mathbb{k}}$ $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$	$(2, 4)$ $(2, 2, 2)$
\mathbb{K}_7 $\pi = 1$ $\pi = -1$	$\langle \rho, \tau, \sigma^2 \rangle$ $\langle \rho, \sigma\tau, \tau^2 \rangle$ $\langle \sigma\rho, \sigma\tau \rangle$ $\langle \tau, \sigma\rho \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ $N_7 = \kappa_{\mathbb{K}_7/\mathbb{k}}$	$(2, 2, 2)$ $(2, 4)$

To check the tables inputs, we use the following remarks and Lemma 8.

Remark 1. According to Artin symbol properties we get:

- $\sigma = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}} \right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P})} \right) = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_2} \right)$, since $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2$.
- $\tau = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}} \right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})} \right) = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_3} \right)$ or $\left(\frac{L/\mathbb{k}}{\mathcal{H}_2\mathcal{H}_3} \right)$, since $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_1\mathcal{H}_3$ or $\mathcal{H}_2\mathcal{H}_3$.
- $\rho = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1} \right)$.

From Lemmas 3 and 5, we deduce that:

Remark 2. (1) Assume that $\left(\frac{p_1}{p_2} \right) = 1$, so

- (i) If $\left(\frac{\pi_1}{\pi_3} \right) = -1$, then $N(\varepsilon_{p_1 p_2}) = 1$, $n = 1$ and $m \geq 3$. Thus $\rho^2 = \tau^4 \sigma^{2^{m-1}}$ or $\rho^2 = \sigma^{2^{m-1}}$.
- (ii) If $\left(\frac{\pi_1}{\pi_3} \right) = 1$, then
 - (a) If $N(\varepsilon_{p_1 p_2}) = 1$, then $m = 2$ and $n \geq 2$. Thus $\rho^2 = \sigma^2 \tau^{2^{n+1}}$ or $\rho^2 = \sigma^2$.
 - (b) If $N(\varepsilon_{p_1 p_2}) = -1$, then $m = 2$ and $n \geq 2$. Thus $\rho^2 = \sigma^2 \tau^{2^n}$ and $\sigma^4 = \tau^{2^{n+1}}$.

(2) Assume that $\left(\frac{p_1}{p_2} \right) = -1$, so $n = 1$, $m \geq 2$, $\rho^2 = \sigma^{2^{m-1}} \tau^2$ and $\sigma^4 = \tau^4$.

Recall that the Artin map ϕ induces the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Cl}_2(\mathbb{k}) & \xrightarrow{\phi} & G/G' \\ J_{\mathbb{K}_j/\mathbb{k}} \downarrow & & \downarrow V_{G/G_j} \\ \mathrm{Cl}_2(\mathbb{K}_j) & \xrightarrow{\phi} & G_j/G'_j \end{array}$$

the rows are isomorphisms and $V_{G/G_j} : G/G' \longrightarrow G_j/G'_j$ is the group transfer map (Verlagerung) which has the following simple characterization when G_j is of index 2 in G . Let $G = G_j \cup zG_j$, then

$$V_{G/G_j}(gG') = \begin{cases} gz^{-1}gz.G'_j = g^2[g, z].G'_j & \text{if } g \in G_j, \\ g^2G'_j & \text{if } g \notin G_j. \end{cases}$$

Thus $\kappa_{\mathbb{K}_j/\mathbb{k}} = \ker J_{\mathbb{K}_j/\mathbb{k}}$ is determined by $\ker V_{G/G_j}$.

Let us show the table inputs for a few examples.

(a) For the extension \mathbb{K}_1 , Table 1 yields that:

$$N_1 = \begin{cases} \langle [\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_2] \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1] \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = -1. \end{cases}$$

$$\text{Hence } G_1 = \text{Gal}(L/\mathbb{K}_1) = \begin{cases} \langle \sigma, \tau\rho, G' \rangle = \langle \sigma, \tau\rho, \tau^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ \langle \sigma, \rho, G' \rangle = \langle \sigma, \rho, \tau^2 \rangle = \langle \sigma, \rho \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = -1. \end{cases}$$

Thus $G/G_1 = \langle \tau \rangle = \{1, \tau G_1\}$. Moreover, Proposition 1 implies that

$$[\tau\rho, \sigma] = [\rho, \sigma] = \sigma^2 \text{ and } [\tau\rho, \tau^2] = [\rho, \tau^2] = \tau^4;$$

$$\text{so } G'_1 = \begin{cases} \langle \tau^4, \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ \langle \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = -1; \end{cases} \text{ this in turn implies that}$$

$$\text{Cl}_2(\mathbb{K}_1) = G_1/G'_1 \simeq \begin{cases} (2, 2, 2) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ (2, 4) & \text{if } \left(\frac{p_1}{p_2}\right) = -1, \end{cases} \text{ since } (\tau\rho)^2, \rho^2 \in G'_1.$$

Compute now the kernel of $V_{G \rightarrow G_1}$.

$$* V_{G \rightarrow G_1}(\sigma G') = \sigma^2 G'_1 = G'_1.$$

$$* V_{G \rightarrow G_1}(\tau G') = \tau^2 G'_1 \neq G'_1.$$

$$* V_{G \rightarrow G_1}(\rho G') = \rho^2 G'_1 = G'_1.$$

Hence $\ker V_{G \rightarrow G_1} = \langle \sigma G', \rho G' \rangle$, thus $\kappa_{\mathbb{K}_1/\mathbb{k}} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1] \rangle = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.

(b) Take another example $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3})$ and assume that $\left(\frac{p_1}{p_2}\right) = -1$, then $n = 1$, $m \geq 2$ and $N(\varepsilon_{p_1 p_2}) = -1$. We have two cases to discuss:

1st case: $\left(\frac{\pi_1}{\pi_3}\right) = -1$. Table 1 yields that $N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle$, hence $G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \tau, \rho, G' \rangle = \langle \tau, \rho \rangle$ or $\langle \sigma\tau, \rho, G' \rangle = \langle \sigma\tau, \rho \rangle$ according as $\tau = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_3}\right)$ or $\left(\frac{L/\mathbb{k}}{\mathcal{H}_2\mathcal{H}_3}\right)$. Thus $G/G_4 = \langle \sigma \rangle = \{1, \sigma G_4\}$. Moreover, Proposition 1 implies that $[\rho, \sigma\tau] = (\sigma\tau)^2$ and $[\rho, \tau] = \tau^2$.

So $G'_4 = \langle \tau^2 \rangle$ or $\langle (\sigma\tau)^2 \rangle$. Therefore $\text{Cl}_2(\mathbb{K}_4) \simeq G_4/G'_4 \simeq (2, 4)$.

Compute $\ker V_{G \rightarrow G_4}$, according as $\tau = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_3}\right)$ or $\left(\frac{L/\mathbb{k}}{\mathcal{H}_2\mathcal{H}_3}\right)$ we get

$$\begin{cases} *V_{G \rightarrow G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\tau G') = \tau^2 [\sigma, \tau] G'_4 = G'_4. \\ *V_{G \rightarrow G_4}(\rho G') = \rho^2 G'_4 = G'_4. \end{cases} \text{ or } \begin{cases} *V_{G \rightarrow G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\tau G') = \tau^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\rho G') = \rho^2 G'_4 = G'_4. \\ *V_{G \rightarrow G_4}(\sigma\tau G') = (\sigma\tau)^2 G'_4 = G'_4. \end{cases}$$

Hence $\ker V_{G \rightarrow G_4} = \langle \tau G', \rho G' \rangle$ or $\langle \sigma\tau G', \rho G' \rangle$, thus

$$\kappa_{\mathbb{K}_4/\mathbb{k}} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_1] \rangle = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle.$$

2nd case: $\left(\frac{\pi_1}{\pi_3}\right) = 1$. Table 1 yields that $N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_1\mathcal{H}_3] \rangle$, hence $G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \tau, \sigma\rho, G' \rangle = \langle \tau, \sigma\rho, \sigma^2 \rangle$ or $\langle \sigma\tau, \tau\rho, G' \rangle = \langle \sigma\tau, \tau\rho, \sigma^2 \rangle$ according as $\tau = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_3}\right)$ or $\left(\frac{L/\mathbb{k}}{\mathcal{H}_2\mathcal{H}_3}\right)$. Thus $G/G_4 = \langle \sigma \rangle = \{1, \sigma G_4\}$. Moreover, Proposition 1 implies that $[\tau\rho, \sigma\tau] = (\sigma\tau)^2$, $[\tau\rho, \sigma^2] = \sigma^4$, $[\sigma\rho, \sigma^2] = \sigma^4$, and $[\sigma\rho, \tau] = \tau^2$. So $G'_4 = \langle \tau^2, \sigma^4 \rangle$ or $\langle \tau^4, (\sigma\tau)^2 \rangle$. Therefore $\text{Cl}_2(\mathbb{K}_4) \simeq G_4/G'_4 \simeq (2, 2, 2)$.

Compute $\ker V_{G \rightarrow G_1}$, according as $\tau = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_3}\right)$ or $\left(\frac{L/\mathbb{k}}{\mathcal{H}_2\mathcal{H}_3}\right)$ we get

$$\left\{ \begin{array}{l} *V_{G \rightarrow G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\tau G') = \tau^2 G'_4 = G'_4. \\ *V_{G \rightarrow G_4}(\rho G') = \rho^2 G'_4 = G'_4. \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} *V_{G \rightarrow G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\tau G') = \tau^2 G'_4 \neq G'_4. \\ *V_{G \rightarrow G_4}(\rho G') = \rho^2 G'_4 = G'_4. \\ *V_{G \rightarrow G_4}(\sigma\tau G') = (\sigma\tau)^2 G'_4 = G'_4. \end{array} \right.$$

Hence $\ker V_{G \rightarrow G_4} = \langle \tau G', \rho G' \rangle$ or $\langle \sigma\tau G', \rho G' \rangle$, thus

$$\kappa_{\mathbb{K}_4/\mathbb{k}} = \langle [\mathcal{H}_1], [\mathcal{H}_3] \rangle.$$

Proceeding similarly, we show the other tables inputs.

5.2.3. Capitulations kernels $\kappa_{\mathbb{L}_j/\mathbb{k}}$ and $\text{Cl}_2(\mathbb{L}_j)$. From the subsection 5.2.2, we deduce that $\kappa_{\mathbb{L}_j/\mathbb{k}} = \text{Cl}_2(\mathbb{k})$. In what follows, we compute the Galois groups $\mathcal{G}_j = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{L}_j)$, their derived groups \mathcal{G}'_j and the abelian type invariants of $\text{Cl}_2(\mathbb{L}_j)$. The results are summarized in the following Tables 4 and 5. Keep the

notations of the Subsection 5.2.2 and put: $\begin{cases} a = \min(n, m), \\ b = \max(n+1, m+1). \end{cases}$

Hence in the Table 4, the left hand sides (if they exist) of the columns \mathcal{G}_j and $\text{Cl}_2(\mathbb{L}_j)$ refer to the case $\beta = 1$, while the right ones refer to the case $\beta = -1$; and in the Table 5, the left hand side of the column \mathcal{G}_j refers to the case $\text{I} = 1$, while the right one refers to the case $\text{I} = -1$.

Table 4: Invariants of $\text{Cl}_2(\mathbb{L}_j)$ for the case $\left(\frac{p_1}{p_2}\right) = 1$

\mathbb{L}_j	\mathcal{G}_j	\mathcal{G}'_j	$\text{Cl}_2(\mathbb{L}_j)$
\mathbb{L}_1	$\langle \tau^2, \sigma \rangle$	$\langle 1 \rangle$	$(2^a, 2^b)$ if $N = -1$ $(2^m, 2^{n+1})$ if $N = 1$
\mathbb{L}_2	$\pi = -1$ $\pi = 1$	$\langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle$ $\langle \tau\rho, \sigma^2, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $\langle \tau^4 \rangle$
\mathbb{L}_3	$\pi = -1$ $\pi = 1$	$\langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle$ $\langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $\langle \tau^4 \rangle$

\mathbb{L}_j		\mathcal{G}_j	\mathcal{G}_j	$\mathcal{Cl}_2(\mathbb{L}_j)$
\mathbb{L}_4	$\pi = -1$	$\langle \sigma \rho, \sigma^2, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$	$(2, 2, 2)$
	$\pi = 1$	$\langle \rho, \tau^2 \rangle$	$\langle \tau^4 \rangle$	$(2, 4)$
\mathbb{L}_5	$\pi = -1$	$\langle \rho, \sigma^2, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$	$(2, 2, 2)$
	$\pi = 1$	$\langle \rho \sigma, \tau^2 \rangle$	$\langle \tau^4 \rangle$	$(2, 4)$
\mathbb{L}_6	$\pi = -1$	$\langle \tau, \sigma^2 \rangle \langle \sigma \tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$(2^{m-1}, 2^{n+2}) (2^m, 2^{n+1})$
	$\pi = 1$			$(2, 2^{n+2})$
\mathbb{L}_7	$\pi = -1$	$\langle \sigma \tau, \sigma^2 \rangle \langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$(2^m, 2^{n+1}) (2^{m-1}, 2^{n+1})$
	$\pi = 1$			$(2, 2^{n+2})$

 Table 5: Invariants of $\mathbf{Cl}_2(\mathbb{L}_j)$ for the case $\left(\frac{p_1}{p_2}\right) = -1$

\mathbb{L}_j	\mathcal{G}_j	\mathcal{G}'_j	$\mathbf{Cl}_2(\mathbb{L}_j)$
\mathbb{L}_1	$\langle \tau^2, \sigma \rangle$	$\langle 1 \rangle$	$(2^a, 2^b)$ if $N = -1$ $(2^m, 2^{n+1})$ if $N = 1$
\mathbb{L}_2 $\pi = 1$ $\pi = -1$	$\langle \sigma\rho, \tau^2 \rangle$ $\langle \rho, \tau^2 \rangle$	$\langle \tau^4 \rangle$	$(2, 4)$
\mathbb{L}_3 $\pi = 1$ $\pi = -1$	$\langle \rho, \tau^2 \rangle$ $\langle \sigma\rho, \tau^2 \rangle$	$\langle \tau^4 \rangle$	$(2, 4)$
\mathbb{L}_4	$\langle \sigma\tau\rho, \sigma^2 \rangle \langle \tau\rho, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_5	$\langle \tau\rho, \sigma^2 \rangle \langle \sigma\tau\rho, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_6 $\pi = 1$ $\pi = -1$	$\langle \tau, \sigma^2 \rangle \langle \sigma\tau, \tau^2 \rangle$ $\langle \sigma\tau, \tau^2 \rangle \langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$ $\langle 1 \rangle$	$(2^2, 2^m)$ $(2, 2^{m+1})$
\mathbb{L}_7 $\pi = 1$ $\pi = -1$	$\langle \sigma\tau, \tau^2 \rangle \langle \tau, \sigma^2 \rangle$ $\langle \tau, \sigma^2 \rangle \langle \sigma\tau, \tau^2 \rangle$	$\langle 1 \rangle$ $\langle 1 \rangle$	$(2, 2^{m+1})$ $(2^2, 2^m)$

Check the entries in some cases.

* Take $\mathbb{L}_1 = \mathbb{K}^{(*)} = \mathbb{K}_1.\mathbb{K}_2.\mathbb{K}_3$. Since $\text{Gal}(L/\mathbb{L}_1) = \mathcal{G}_1 = G_1 \cap G_2$, then

$\mathcal{G}_1 = \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma, \rho, \tau^2 \rangle = \langle \sigma, \tau^2 \rangle$, thus $\mathcal{G}'_1 = \langle 1 \rangle$. As

$$\begin{cases} \sigma^{2^m} = \tau^{2^{n+1}} = 1 & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ \sigma^{2^m} = \tau^{2^{n+1}} \text{ and } \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1 & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \end{cases}$$

$$\text{so } \mathbf{Cl}_2(\mathbb{L}_1) \simeq \begin{cases} (2^{n+1}, 2^m) & \text{if } N(\varepsilon_{p_1 p_2}) = 1, \\ (2^{\min(n,m)}, 2^{\max(n+1,m+1)}) & \text{if } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

* Take $\mathbb{L}_2 = \mathbb{K}_1.\mathbb{K}_4.\mathbb{K}_6$ and assume that $\left(\frac{p_1}{p_2}\right) = 1$, then $\mathcal{G}_2 = \text{Gal}(L/\mathbb{L}_2) = G_1 \cap G_4 \cap G_6$. There are two cases to distinguish:

- 1st case: If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\mathcal{G}_2 = \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \rho, \tau^2 \rangle = \langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle$ or $\langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \tau\rho, \sigma^2 \rangle = \langle \tau\rho, \sigma^2, \tau^2 \rangle$ according as $\beta = 1$ or -1 .

Thus $\mathcal{G}'_2 = \langle \sigma^4, \tau^4 \rangle$. On the other hand, in this case we have $(\sigma\tau\rho)^2 = (\tau\rho)^2 = \rho^2\tau^{2^{n+1}}$, so $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 2, 2)$.

- 2nd case: If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = 1$, then

$\mathcal{G}_2 = \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \tau, \rho, \sigma^2 \rangle = \langle \tau\rho, \sigma^2, \tau^2 \rangle$ or $\langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \rho, \sigma^2 \rangle = \langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle$ according as $\beta = 1$ or -1 . As in this case

$$(\tau\rho)^2 = (\sigma\tau\rho)^2 = \begin{cases} \rho^2 & \text{if } N(\varepsilon_{p_1 p_2}) = -1, \\ \rho^2\tau^{2^{n+1}} & \text{if } N(\varepsilon_{p_1 p_2}) = 1; \end{cases}$$

so $\mathcal{G}_2 = \langle \tau\rho, \tau^2 \rangle$ or $\langle \sigma\tau\rho, \tau^2 \rangle$. We infer that $\mathcal{G}'_2 = \langle \tau^4 \rangle$. From which we deduce that $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 4)$, since $(\sigma\tau\rho)^4 = (\tau\rho)^4 = 1$.

Assume now that $\left(\frac{p_1}{p_2}\right) = -1$, then $\mathcal{G}_2 = \begin{cases} \langle \rho, \tau^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \\ \langle \sigma\rho, \tau^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1. \end{cases}$

Thus $\mathcal{G}'_2 = \langle \tau^4 \rangle$, hence $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 4)$.

* Take $\mathbb{L}_6 = \mathbb{K}_3.\mathbb{K}_4.\mathbb{K}_7$ and assume that $\left(\frac{p_1}{p_2}\right) = -1$, then $\mathcal{G}_6 = \text{Gal}(L/\mathbb{L}_4) = G_3 \cap G_4 \cap G_7$. There are two cases to distinguish:

• 1st case: If $\left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = -1$, then Lemma 5 implies that $n = 1$ and $m = 2$, hence $\sigma^4 = \tau^4$ and $\rho^2 = \sigma^2\tau^2$. We have also two sub-cases to discuss:

a - If $\left(\frac{\pi_1}{\pi_3}\right) = 1$, then Table 3 yields that $\mathcal{G}_6 = \langle \sigma\tau, \tau^2 \rangle = \langle \sigma\tau, \sigma^2 \rangle$, thus $\mathcal{G}'_6 = \langle 1 \rangle$. As $(\sigma\tau)^4 = \sigma^4\tau^4 = \tau^8 = 1$ and $(\sigma^2)^{2^m} = \sigma^8 = 1$, so $\mathbf{Cl}_2(\mathbb{L}_6) \simeq (2^2, 2^m)$.

b - If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then Table 3 yields that $\mathcal{G}_6 = \langle \sigma^2, \tau \rangle$, thus $\mathcal{G}'_6 = \langle 1 \rangle$. As $(\sigma^2)^2 = \tau^4$ and $\tau^{2^{n+2}} = \tau^8 = 1$, so $\mathbf{Cl}_2(\mathbb{L}_6) \simeq (2, 2^{m+1})$.

• 2nd case: If $\left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = 1$, then Lemma 5 implies that $n = 1$ and $m \geq 3$, hence $\sigma^{2^m} = \tau^4$ and $\rho^2 = \tau^2\sigma^{2^{m-1}}$. We have also two sub-cases to discuss:

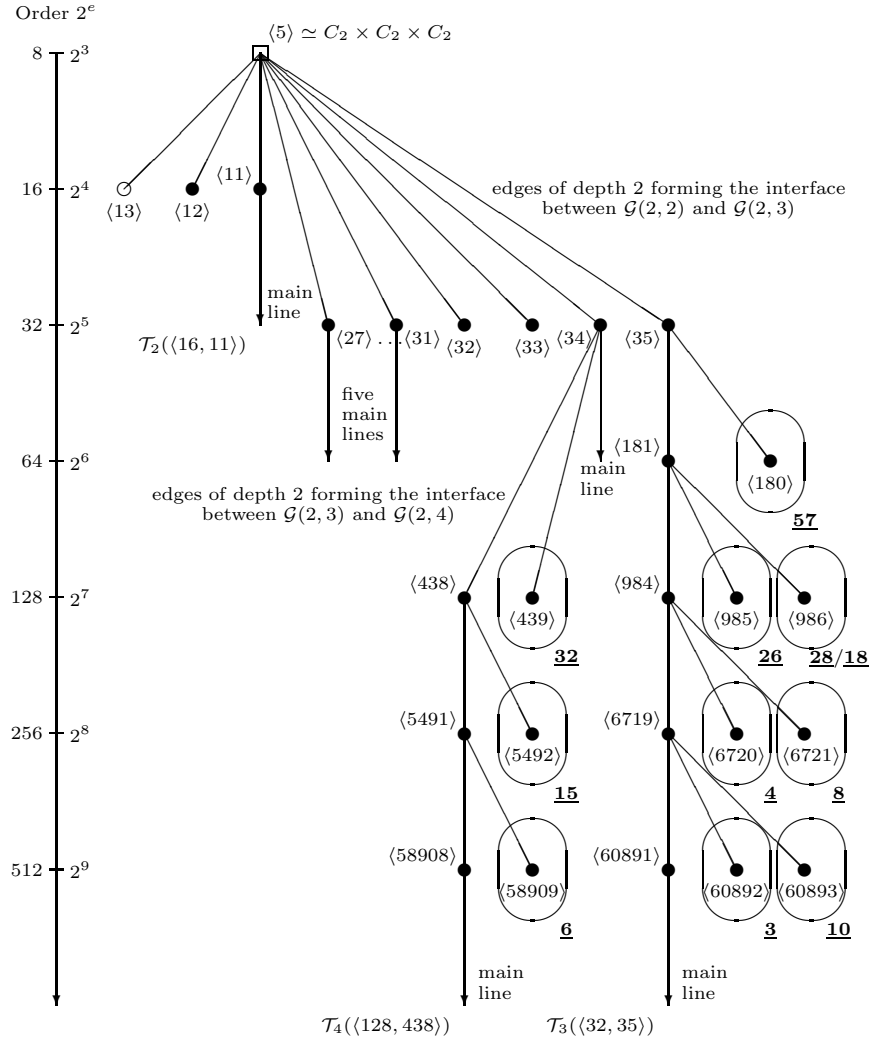
a - If $\left(\frac{\pi_1}{\pi_3}\right) = 1$, then Table 3 yields that $\mathcal{G}_6 = \langle \tau, \sigma^2 \rangle$, thus $\mathcal{G}'_6 = \langle 1 \rangle$. As $\tau^{2^2} = \tau^4 = \sigma^{2^m} = (\sigma^2)^{2^{m-1}}$ and $(\sigma^2)^{2^m} = \sigma^{2^{m+1}} = 1$, so $\mathbf{Cl}_2(\mathbb{L}_6) \simeq (2^2, 2^m)$.

b - If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then Table 3 yields that $\mathcal{G}_6 = \langle \sigma\tau, \tau^2 \rangle$, thus $\mathcal{G}'_6 = \langle 1 \rangle$. As $(\tau^2)^2 = \tau^4 = \sigma^{2^m} = \sigma^{2^m}\tau^{2^m} = (\sigma\tau)^{2^m}$ and $\sigma\tau$ is of order 2^{m+1} , so $\mathbf{Cl}_2(\mathbb{L}_6) \simeq (2, 2^{m+1})$. The other tables entries are checked similarly.

6. NUMERICAL EXAMPLES

To obtain a first impression of how the 2-tower groups $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ are distributed on the coclass graphs $\mathcal{G}(2, r)$ for $3 \leq r \leq 4$ [20, § 2, p. 410], we have analyzed the 207 bicyclic biquadratic fields $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ with $d = p_1 p_2 q < 50000$. All computations were done with the aid of PARI/GP [24]. Using the

presentations in terms of the parameters m , n , and $N = N(\varepsilon_{p_1 p_2})$ in Theorem 2, we can identify these Galois groups G of the maximal unramified pro-2 extensions $\mathbb{k}_2^{(2)}$ of bicyclic biquadratic fields \mathbb{k} by means of the SmallGroups Library [11] by Besche, Eick, and O'Brien. It turns out that most of the occurring groups are members of *one-parameter families*, which can be identified with infinite *periodic sequences* in the sense of [20, p. 411]. For this identification it is convenient to give *polycyclic presentations* of the groups which are particularly adequate for emphasizing the structure of the lower central series of G and of the coclass tree where G is located.

FIGURE 5. Relevant part of the descendant tree of $C_2 \times C_2 \times C_2$ 

The descendant tree of the elementary abelian 2-group of rank 3 is drawn for the first time in Figure 5. Vertices G of this diagram are classified according to their centre $\zeta(G)$ by using different symbols:

- (1) a large contour square \square denotes an abelian group,
- (2) a big contour circle \circ represents a metabelian group with cyclic centre of order 4, and
- (3) big full discs \bullet represent metabelian groups with bicyclic centre of type $(2, 2)$.

Groups are labelled by a number in angles which is the identifier in the Small-Groups Library [11]. Here, we omit the order which is given on the left hand scale. The actual distribution of the 207 second 2-class groups G of bicyclic biquadratic fields $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ of type $(2, 2, 2)$ with radicand $0 < d = p_1 p_2 q < 50000$ is represented by underlined boldface counters (absolute frequencies) of hits of the vertex surrounded by the adjacent oval.

6.1. The tree $\mathcal{T}_3(\langle 32, 35 \rangle)$ of the coclass graph $\mathcal{G}(2, 3)$. Without exceptions, all groups G of coclass 3 are vertices of the coclass tree with root $\langle 32, 35 \rangle$. They can be partitioned into four categories. Firstly, the *pre-periodic* vertex $\langle 64, 180 \rangle$ on branch 1 of this tree. The associated fields have parameters $m = 2$, $n = 1$, $N = -1$ and $(\frac{p_1}{p_2}) = +1$. Periodicity sets in with branch 2 and gives rise to *two periodic sequences* which form the second and third category. A periodic sequence always consists of a ground state and infinitely many higher excited states [20, p. 424]. Finally, there is a *variant* of the vertex $\langle 128, 986 \rangle$, exceptional from the number theoretic point of view, since the maximal subgroups of G are associated to the unramified quadratic extensions of \mathbb{k} in a different manner than for the group-theoretically isomorphic ground state of the second periodic sequence.

Example 1. The most frequent group $G \simeq \langle 64, 180 \rangle$ with parameters $m = 2$, $n = 1$ occurs 57 times (28%) for the radicands $d \in \{455, 795, 2955, \dots\}$ with $N = -1$ and $(\frac{p_1}{p_2}) = +1$. It is a terminal leaf on the first branch of the coclass tree $\mathcal{T}_3(\langle 32, 35 \rangle)$, whose mainline vertices [20, pp. 410–411] arise as finite quotients of the infinite topological pro-2 group given by Newman and O’Brien in [22, App. A, no. 79, p. 155, and App. B, Tbl. 79, p. 168]. They form the infinite periodic sequence $(\langle 32, 35 \rangle, \langle 64, 181 \rangle, \langle 128, 984 \rangle, \langle 256, 6719 \rangle, \langle 512, 60891 \rangle, \dots)$ with parametrized pc-presentation $(n \geq 2)$

$$\begin{aligned} \langle x, y, z \mid s_2 = [y, x], \ t_2 = [z, x], \ t_j = [t_{j-1}, x] \text{ for } 3 \leq j \leq n+2, \\ x^2 = s_2, \ y^2 = s_2, \ z^2 = t_2 t_3, \ s_2^2 = 1, \ t_j^2 = t_{j+1} t_{j+2} \text{ for } 2 \leq j \leq n, \ t_{n+1}^2 = t_{n+2} \rangle. \end{aligned}$$

Example 2. (Second 2-class groups G with parameters $m = 2$, $n \geq 2$, $N = -1$)
The groups of class $c(G) = n + 2$,

$$G = \langle \rho, \sigma, \tau \mid \rho^4 = \sigma^8 = \tau^{2^{n+2}} = 1, \rho^2 = \tau^{2^n} \sigma^2, \sigma^4 = \tau^{2^{n+1}}, [\sigma, \tau] = 1, [\sigma, \rho] = \sigma^2, [\rho, \tau] = \tau^2 \rangle,$$

with $n \geq 2$, form the infinite periodic sequence ($\langle 128, 985 \rangle$, $\langle 256, 6720 \rangle$, $\langle 512, 60892 \rangle$, ...) with parametrized pc-presentation

$$\begin{aligned} \langle x, y, z \mid s_2 = [y, x], t_2 = [z, x], t_j = [t_{j-1}, x] \text{ for } 3 \leq j \leq n+2, \\ x^2 = s_2, y^2 = s_2 t_{n+2}, z^2 = t_2 t_3, s_2^2 = 1, t_j^2 = t_{j+1} t_{j+2} \text{ for } 2 \leq j \leq n, t_{n+1}^2 = t_{n+2} \rangle. \end{aligned}$$

The ground state $\langle 128, 985 \rangle$ with $n = 2$ occurs 26 times (13%) for $d \in \{435, 6235, 6815, \dots\}$, the first excited state $\langle 256, 6720 \rangle$ with $n = 3$ occurs 4 times for $d \in \{15915, 17139, 42915, 46587\}$, and the second excited state $\langle 512, 60892 \rangle$ with $n = 4$ occurs 3 times for $d \in \{6915, 16135, 39315\}$.

Example 3. (Second 2-class groups G with parameters $m \geq 3$, $n = 1$, $N = -1$)
The groups of class $c(G) = m + 1$,

$$G = \langle \rho, \sigma, \tau \mid \rho^4 = \sigma^{2^{m+1}} = \tau^8 = 1, \rho^2 = \tau^2 \sigma^{2^{m-1}}, \sigma^{2^m} = \tau^4, [\sigma, \tau] = 1, [\sigma, \rho] = \sigma^{2^{m-2}}, [\rho, \tau] = \tau^2 \rangle,$$

with $m \geq 3$, form the infinite periodic sequence ($\langle 128, 986 \rangle$, $\langle 256, 6721 \rangle$, $\langle 512, 60893 \rangle$, ...) with parametrized pc-presentation

$$\begin{aligned} \langle x, y, z \mid s_2 = [y, x], t_2 = [z, x], t_j = [t_{j-1}, x] \text{ for } 3 \leq j \leq m+1, \\ x^2 = s_2, y^2 = s_2, z^2 = t_2 t_3 t_{m+1}, s_2^2 = 1, t_j^2 = t_{j+1} t_{j+2} \text{ for } 2 \leq j \leq m-1, t_m^2 = t_{m+1} \rangle. \end{aligned}$$

The ground state $\langle 128, 986 \rangle$ with $m = 3$ occurs 28 times (14%) for $d \in \{2595, 4255, 4395, \dots\}$, the first excited state $\langle 256, 6721 \rangle$ with $m = 4$ occurs 8 times for $d \in \{8355, 19155, 24195, \dots\}$, and the second excited state $\langle 512, 60893 \rangle$ with $m = 5$ occurs 10 times for $d \in \{19459, 26663, 28171, \dots\}$. We also observed a third excited state $\langle 512, 60891 \rangle - \#1; 3$ with $m = 6$ for a single radicand $d = 79651$ outside of the range of our systematic investigations.

We point out again that the group $\langle 64, 180 \rangle$ with $m = 2$, $n = 1$, $N = -1$ is pre-periodic and its presentation does not fit into either of the last two examples, whence it does not belong to either of the two mentioned periodic non-mainline sequences.

The last three examples suggest a very promising *characterization of mainline vertices* on coclass trees of arbitrary p -groups, which seems to be of a fairly general nature and obviously has not been recognized by other investigators up to now.

Remark 3. Mainline Principle: The relators for p th powers of generators of mainline groups are distinguished by being independent of the class, whereas a

relator of any non-mainline group contains the generator of the last non-trivial term of the lower central series as a *small perturbation*.

In Example 2, resp. 3, the small perturbation is the generator $t_{n+2} \in \gamma_{n+2}(G)$, resp. $t_{m+1} \in \gamma_{m+1}(G)$, of the last non-trivial lower central in the relation $y^2 = s_2 t_{n+2}$, where $n+2 = c(G)$, resp. $z^2 = t_2 t_3 t_{m+1}$, where $m+1 = c(G)$. For the rest, the presentation coincides with the mainline presentation.

Example 4. There exists a unique group which can be characterized by two distinct couples of parameters (m, n) . Whereas $m = n = 2$ with $N = -1$ gives rise to $\langle 128, 985 \rangle$, the same parameters $m = n = 2$ with $N = +1$ define a *variant* of $\langle 128, 986 \rangle$, which was given by $m = 3, n = 1, N = -1$ already. This variant is realized 18 times (9%) for the radicands $d \in \{1515, 3535, 5551, \dots\}$.

6.2. The tree $\mathcal{T}_4(\langle 128, 438 \rangle)$ of the coclass graph $\mathcal{G}(2, 4)$. The groups G of coclass 4 are either vertices of the coclass tree with root $\langle 128, 438 \rangle$ or they populate the single sporadic *isolated* vertex $\langle 128, 439 \rangle$ outside of any coclass trees. The associated fields are characterized by $N = +1$ and $(\frac{p_1}{p_2}) = +1$, according to Theorem 2. Periodicity sets in with branch 1 already and gives rise to a *single periodic non-mainline sequence*. The mainline of this tree is given by the parametrized pc-presentation ($m \geq 3$)

$$\begin{aligned} \langle x, y, z \mid s_2 = [y, x], t_2 = [z, x], s_j = [s_{j-1}, x] \text{ for } 3 \leq j \leq m, t_3 = [t_2, x], \\ x^2 = 1, y^2 = s_2 s_3, z^2 = t_2, s_j^2 = s_{j+1} s_{j+2} \text{ for } 2 \leq j \leq m-2, s_{m-1}^2 = s_m, t_2^2 = t_3 \rangle. \end{aligned}$$

Example 5. With 32 occurrences (15%) the second largest density of population arises for the *sporadic isolated vertex* $\langle 128, 439 \rangle$ with parameters $m = 3, n = 1, N = +1$, which occurs for $d \in \{2135, 2235, 4035, \dots\}$. The realm of coclass $cc(G) = 4$ commences with this immediate descendant of *depth two* of $\langle 32, 34 \rangle$, called an *interface group* at the border of the coclass graphs $\mathcal{G}(2, 3)$ and $\mathcal{G}(2, 4)$ in [20, pp. 430–434]. It is an isolated top-vertex of the sporadic part of $\mathcal{G}(2, 4)$, lying outside of any coclass trees. In particular, it has nothing to do with the coclass tree of $\langle 64, 174 \rangle$ arising from its generalized parent $\langle 32, 34 \rangle$, whose projective mainline limit is given by [22, App. A, no. 78, p. 155, and App. B, Tbl. 78, p. 167].

Example 6. (Second 2-class groups G with parameters $m \geq 4, n = 1, N = +1$) The groups of class $c(G) = m$,

$$G = \langle \rho, \sigma, \tau \mid \rho^4 = \sigma^{2^m} = \tau^8 = 1, \rho^2 = \tau^4 \sigma^{2^{m-1}}, [\sigma, \tau] = 1, [\rho, \sigma] = \sigma^2, [\tau, \rho] = \tau^2 \rangle,$$

with $m \geq 4$, form the infinite periodic sequence $(\langle 256, 5492 \rangle, \langle 512, 58909 \rangle, \dots)$ with parametrized pc-presentation

$$\langle x, y, z \mid s_2 = [y, x], t_2 = [z, x], s_j = [s_{j-1}, x] \text{ for } 3 \leq j \leq m, t_3 = [t_2, x], \\ x^2 = s_m, y^2 = s_2 s_3, z^2 = t_2, s_j^2 = s_{j+1} s_{j+2} \text{ for } 2 \leq j \leq m-2, s_{m-1}^2 = s_m, t_2^2 = t_3 \rangle.$$

The group $\langle 128, 439 \rangle$ with $m = 3$, $n = 1$, $N = +1$ is sporadic and even isolated, but its presentation is also of the same form. The ground state $\langle 256, 5492 \rangle$ with $m = 4$ occurs 15 times (7%) for $d \in \{10515, 12535, 12963, \dots\}$, the first excited state $\langle 512, 58909 \rangle$ with $m = 5$ occurs 6 times for $d \in \{34635, 41115, 41835, \dots\}$.

We conclude this section with the following tables: Table 6 gives the structure of the class group $\mathbf{Cl}(\mathbb{k})$ of the bicyclic biquadratic field \mathbb{k} , its discriminant $\text{disc}(\mathbb{k})$, the structure of the class groups of its two quadratic subfields k_0 and \bar{k}_0 , and the coclass of G . Tables 7 and 8, resp. 9 and 10, give the structure of the class groups $\mathbf{Cl}(\mathbb{K}_j)$, resp. $\mathbf{Cl}(\mathbb{L}_j)$, for the case $(\frac{p_1}{p_2}) = 1$. Finally, Tables 11 and 12, resp. 13, give the structure of the class groups $\mathbf{Cl}(\mathbb{K}_j)$, resp. $\mathbf{Cl}(\mathbb{L}_j)$, for the case $(\frac{p_1}{p_2}) = -1$. We briefly put $N = N(\varepsilon_{p_1 p_2})$, $\gamma = \left(\frac{p_1}{p_2}\right)_4$ and $\delta = \left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4$.

Table 6: Invariants of \mathbb{k}

$d = p_1.p_2.q$	γ	δ	N	m, n	$\mathbf{Cl}(k_0)$	$\mathbf{Cl}(\bar{k}_0)$	$\mathbf{Cl}(\mathbb{k})$	$\text{disc}(\mathbb{k})$	$cc(G)$
435 = 5.29.3	1	1	-1	2, 2	(2, 2)	(2, 2)	(2, 2, 2)	3027600	3
455 = 5.13.7	-1		-1	2, 1	(2, 2)	(10, 2)	(10, 2, 2)	3312400	3
795 = 5.53.3	-1		-1	2, 1	(2, 2)	(2, 2)	(2, 2, 2)	10112400	3
1515 = 5.101.3	1	1	1	2, 2	(2, 2)	(6, 2)	(6, 2, 2)	36723600	3
2135 = 5.61.7	1	-1	1	3, 1	(2, 2)	(22, 2)	(22, 2, 2)	72931600	4
2235 = 5.149.3	1	-1	1	3, 1	(2, 2)	(6, 2)	(6, 2, 2)	79923600	4
2595 = 5.173.3	-1		-1	3, 1	(2, 2)	(6, 2)	(6, 2, 2)	107744400	3
2955 = 5.197.3	-1		-1	2, 1	(2, 2)	(6, 2)	(6, 2, 2)	139712400	3
3055 = 5.13.47	-1		-1	2, 1	(2, 2)	(18, 2)	(18, 2, 2)	149328400	3
3535 = 5.101.7	1	1	1	2, 2	(2, 2)	(14, 2)	(14, 2, 2)	199939600	3
5551 = 13.61.7	1	1	1	2, 2	(2, 2)	(26, 2)	(26, 2, 2)	493017616	3
5835 = 5.389.3	1	-1	1	3, 1	(2, 2)	(6, 2)	(6, 2, 2)	544755600	4
7015 = 5.61.23	1	-1	1	3, 1	(2, 2)	(14, 2)	(14, 2, 2)	787363600	4
7163 = 13.29.19	1	-1	1	3, 1	(2, 2)	(10, 2)	(10, 2, 2)	820937104	4
9415 = 5.269.7	1	-1	1	3, 1	(2, 2)	(26, 2)	(26, 2, 2)	1418275600	4
10515 = 5.701.3	1	-1	1	4, 1	(2, 2)	(10, 2)	(10, 2, 2)	1769043600	4

Table 7: Invariants of \mathbb{K}_j for the case $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{\pi_1}{\pi_3}\right) = -1$

$d = p_1.p_2.q$	m, n	$\mathbf{Cl}(\mathbb{K}_1)$	$\mathbf{Cl}(\mathbb{K}_2)$	$\mathbf{Cl}(\mathbb{K}_3)$	$\mathbf{Cl}(\mathbb{K}_4)$	$\mathbf{Cl}(\mathbb{K}_5)$	$\mathbf{Cl}(\mathbb{K}_6)$	$\mathbf{Cl}(\mathbb{K}_7)$
2135 = 5.61.7	3, 1	(66, 2, 2)	(66, 2, 2)	(88, 8)	(22, 2, 2)	(22, 2, 2)	(22, 2, 2)	(22, 2, 2)
2235 = 5.149.3	3, 1	(42, 2, 2)	(42, 2, 2)	(24, 8)	(6, 2, 2)	(6, 6, 2)	(6, 6, 2)	(6, 2, 2)
4035 = 5.269.3	3, 1	(42, 2, 2)	(66, 2, 2)	(24, 24)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)
4147 = 13.29.11	3, 1	(30, 2, 2)	(30, 6, 2)	(24, 8)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)

$d = p_1.p_2.q$	m, n	$Cl(\mathbb{K}_1)$	$Cl(\mathbb{K}_2)$	$Cl(\mathbb{K}_3)$	$Cl(\mathbb{K}_4)$	$Cl(\mathbb{K}_5)$	$Cl(\mathbb{K}_6)$	$Cl(\mathbb{K}_7)$
4611 = 29.53.3	3, 1	(210, 2, 2)	(42, 6, 2)	(56, 8)	(70, 2, 2)	(42, 2, 2)	(42, 2, 2)	(70, 2, 2)
5835 = 5.389.3	3, 1	(66, 2, 2)	(66, 2, 2)	(24, 8)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)	(6, 2, 2)
14287 = 13.157.7	4, 1	(18, 6, 2)	(18, 6, 2)	(144, 8)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)
15051 = 29.173.3	4, 1	(126, 6, 2)	(42, 14, 2)	(112, 8)	(14, 2, 2)	(14, 2, 2)	(14, 2, 2)	(14, 2, 2)
17715 = 5.1181.3	4, 1	(18, 18, 2)	(414, 2, 2)	(144, 8)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)
19515 = 5.1301.3	4, 1	(234, 2, 2)	(450, 2, 2)	(144, 8)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)	(18, 2, 2)

Table 8: Invariants of \mathbb{K}_j for the case $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{\pi_1}{\pi_3}\right) = 1$

$d = p_1.p_2.q$	m, n	$Cl(\mathbb{K}_1)$	$Cl(\mathbb{K}_2)$	$Cl(\mathbb{K}_3)$	$Cl(\mathbb{K}_4)$	$Cl(\mathbb{K}_5)$	$Cl(\mathbb{K}_6)$	$Cl(\mathbb{K}_7)$
1515 = 5.101.3	2, 2	(30, 2, 2)	(42, 2, 2)	(48, 4)	(12, 2)	(12, 2)	(12, 2)	(12, 2)
3535 = 5.101.7	2, 2	(42, 2, 2)	(14, 14, 2)	(112, 4)	(28, 2)	(28, 2)	(28, 2)	(28, 2)
5551 = 13.61.7	2, 2	(78, 2, 2)	(78, 2, 2)	(208, 4)	(52, 2)	(52, 2)	(52, 2)	(52, 2)
6235 = 5.29.43	2, 2	(78, 2, 2)	(42, 6, 2)	(48, 4)	(60, 2)	(12, 2)	(12, 2)	(60, 2)
6335 = 5.181.7	2, 2	(138, 2, 2)	(230, 2, 2)	(1104, 4)	(92, 2)	(92, 2)	(92, 2)	(92, 2)
6815 = 5.29.47	2, 2	(138, 2, 2)	(138, 6, 2)	(1840, 4)	(92, 2)	(92, 2)	(92, 2)	(92, 2)
6915 = 5.461.3	2, 4	(126, 2, 2)	(210, 2, 2)	(1344, 4)	(84, 2)	(28, 2)	(28, 2)	(84, 2)
15915 = 5.1061.3	2, 3	(238, 2, 2)	(182, 2, 2)	(1120, 4)	(28, 2)	(28, 2)	(28, 2)	(28, 2)
16135 = 5.461.7	2, 4	(154, 2, 2)	(330, 2, 2)	(2112, 4)	(132, 6)	(44, 2)	(44, 2)	(132, 6)
17139 = 29.197.3	2, 3	(462, 2, 2)	(210, 2, 2)	(672, 4)	(84, 2)	(28, 2)	(28, 2)	(84, 2)

Table 9: Invariants of \mathbb{L}_j for $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{\pi_1}{\pi_3}\right) = 1$

$d = p_1.p_2.q$	m, n	N	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
3535 = 5.101.7	2, 2	1	(168, 28)	(84, 2)	(84, 2)	(28, 14)	(28, 14)	(112, 2)	(112, 2)
6815 = 5.29.47	2, 2	-1	(2760, 12)	(276, 2)	(276, 2)	(276, 6)	(276, 6)	(1840, 2)	(1840, 2)
6915 = 461.5.3	2, 4	-1	(10080, 12)	(420, 6)	(420, 6)	(252, 6)	(252, 6)	(1344, 6)	(1344, 2)
7635 = 5.509.3	2, 2	-1	(840, 60)	(420, 2)	(420, 2)	(60, 10)	(60, 10)	(240, 2)	(240, 2)
8723 = 13.61.11	2, 2	1	(840, 60)	(420, 2)	(420, 2)	(420, 2)	(420, 2)	(112, 2)	(112, 2)
12215 = 5.349.7	2, 2	-1	(19320, 4)	(276, 2)	(276, 2)	(644, 2)	(644, 2)	(1840, 2)	(1840, 2)
15915 = 1061.5.3	2, 3	-1	(123760, 4)	(364, 2)	(364, 2)	(476, 2)	(476, 2)	(1120, 2)	(1120, 2)
16135 = 5.461.7	2, 4	-1	(36960, 12)	(924, 6)	(924, 6)	(660, 6)	(660, 6)	(2112, 6)	(2112, 2)
17139 = 197.29.3	2, 3	-1	(18480, 12)	(420, 6)	(420, 6)	(924, 6)	(924, 6)	(672, 6)	(672, 2)

Table 10: Invariants of \mathbb{L}_j for $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{\pi_1}{\pi_3}\right) = -1$

$d = p_1.p_2.q$	m, n	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
2135 = 5.61.7	3, 1	(264, 12)	(66, 2, 2)	(66, 2, 2)	(66, 2, 2)	(66, 2, 2)	(88, 4)	(88, 4)
2235 = 149.5.3	3, 1	(168, 28)	(42, 6, 2)	(42, 6, 2)	(42, 6, 2)	(42, 6, 2)	(24, 4)	(24, 12)
4035 = 5.269.3	3, 1	(1848, 12)	(42, 2, 2)	(42, 2, 2)	(66, 2, 2)	(66, 2, 2)	(24, 12)	(24, 12)
4147 = 29.13.11	3, 1	(120, 60)	(30, 6, 2)	(30, 6, 2)	(30, 2, 2)	(30, 2, 2)	(24, 4)	(24, 4)
5835 = 389.5.3	3, 1	(264, 44)	(66, 2, 2)	(66, 2, 2)	(66, 2, 2)	(66, 2, 2)	(24, 4)	(24, 4)
7015 = 5.61.23	3, 1	(168, 84)	(70, 14, 2)	(70, 14, 2)	(210, 2, 2)	(210, 2, 2)	(840, 20)	(168, 4)
10515 = 701.5.3	4, 1	(1360, 68)	(170, 2, 2)	(170, 2, 2)	(170, 2, 2)	(170, 2, 2)	(80, 4)	(40, 8)
11687 = 13.29.31	3, 1	(3192, 12)	(798, 2, 2)	(798, 2, 2)	(114, 2, 2)	(114, 2, 2)	(456, 4)	(456, 4)
12315 = 821.5.3	3, 1	(2040, 60)	(30, 30, 2)	(30, 30, 2)	(510, 2, 2)	(510, 2, 2)	(120, 12)	(120, 12)
12963 = 149.29.3	4, 1	(1680, 420)	(210, 2, 2)	(210, 2, 2)	(210, 2, 2)	(210, 2, 2)	(40, 40)	(80, 20)

Table 11: Invariants of \mathbb{K}_j for $\left(\frac{p_1}{p_2}\right) = -1$ and $\left(\frac{\pi_1}{\pi_3}\right) = 1$

$d = p_1 \cdot p_2 \cdot q$	m, n	$Cl(\mathbb{K}_1)$	$Cl(\mathbb{K}_2)$	$Cl(\mathbb{K}_3)$	$Cl(\mathbb{K}_4)$	$Cl(\mathbb{K}_5)$	$Cl(\mathbb{K}_6)$	$Cl(\mathbb{K}_7)$
795 = 5.53.3	2, 1	(20, 2)	(12, 2)	(8, 4)	(2, 2, 2)	(4, 2)	(4, 2)	(2, 2, 2)
2955 = 5.197.3	2, 1	(132, 2)	(60, 2)	(24, 12)	(6, 2, 2)	(12, 2)	(12, 2)	(6, 2, 2)
4755 = 5.317.3	2, 1	(156, 6)	(60, 6)	(24, 12)	(6, 6, 6)	(12, 6)	(12, 6)	(6, 6, 6)
6095 = 5.53.23	2, 1	(84, 6)	(84, 6)	(168, 12)	(42, 2, 2)	(84, 2)	(84, 2)	(42, 2, 2)
8787 = 29.101.3	2, 1	(60, 6)	(84, 6)	(120, 4)	(6, 2, 2)	(12, 2)	(12, 2)	(6, 2, 2)
10255 = 5.293.7	3, 1	(396, 6)	(396, 6)	(528, 4)	(66, 2, 2)	(660, 2)	(660, 2)	(66, 2, 2)
15587 = 13.109.11	3, 1	(684, 2)	(180, 6)	(144, 4)	(18, 2, 2)	(36, 2)	(36, 2)	(18, 2, 2)
19155 = 5.1277.3	4, 1	(252, 6)	(612, 2)	(288, 4)	(18, 2, 2)	(36, 2)	(36, 2)	(18, 2, 2)
24195 = 5.1613.3	4, 1	(1508, 2)	(1092, 2)	(416, 4)	(78, 6, 2)	(52, 2)	(52, 2)	(78, 6, 2)
26663 = 13.293.7	5, 1	(1116, 2)	(1116, 2)	(1984, 4)	(62, 2, 2)	(372, 2)	(372, 2)	(62, 2, 2)

Table 12: Invariants of \mathbb{K}_j for $\left(\frac{p_1}{p_2}\right) = -1$ and $\left(\frac{\pi_1}{\pi_3}\right) = -1$

$d = p_1 \cdot p_2 \cdot q$	m, n	$Cl(\mathbb{K}_1)$	$Cl(\mathbb{K}_2)$	$Cl(\mathbb{K}_3)$	$Cl(\mathbb{K}_4)$	$Cl(\mathbb{K}_5)$	$Cl(\mathbb{K}_6)$	$Cl(\mathbb{K}_7)$
455 = 5.13.7	2, 1	(20, 2)	(20, 2)	(40, 4)	(20, 2)	(10, 2, 2)	(10, 2, 2)	(20, 2)
2595 = 5.173.3	3, 1	(36, 6)	(84, 2)	(48, 4)	(12, 2)	(6, 2, 2)	(6, 2, 2)	(12, 2)
3055 = 5.13.47	2, 1	(180, 2)	(36, 6)	(360, 4)	(468, 2)	(18, 2, 2)	(18, 2, 2)	(468, 2)
4255 = 5.37.23	3, 1	(180, 2)	(36, 2)	(144, 12)	(36, 2)	(18, 2, 2)	(18, 2, 2)	(36, 2)
4355 = 5.13.67	2, 1	(660, 2)	(180, 6)	(120, 4)	(60, 2)	(30, 2, 2)	(30, 2, 2)	(60, 2)
5395 = 5.13.83	2, 1	(204, 2)	(60, 2)	(24, 12)	(12, 2)	(6, 2, 2)	(6, 2, 2)	(12, 2)
5495 = 5.157.7	3, 1	(84, 6)	(84, 6)	(336, 12)	(84, 2)	(42, 2, 2)	(42, 2, 2)	(84, 2)
6055 = 5.173.7	3, 1	(252, 6)	(252, 2)	(144, 4)	(36, 2)	(18, 2, 2)	(18, 2, 2)	(36, 2)
7955 = 5.37.43	3, 1	(44, 22)	(308, 2)	(176, 4)	(44, 2)	(22, 2, 2)	(22, 2, 2)	(44, 2)
8355 = 5.557.3	4, 1	(380, 2)	(180, 2)	(160, 4)	(20, 2)	(10, 2, 2)	(10, 2, 2)	(20, 2)

Table 13: Invariants of \mathbb{L}_j for $\left(\frac{p_1}{p_2}\right) = -1$

$d = p_1 \cdot p_2 \cdot q$	m, n	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
455 = 5.13.7	2, 1	(40, 2)	(20, 2)	(20, 2)	(20, 2)	(20, 2)	(40, 2)	(20, 4)
795 = 5.53.3	2, 1	(120, 2)	(20, 2)	(20, 2)	(12, 2)	(12, 2)	(4, 4)	(8, 2)
3055 = 5.13.47	2, 1	(360, 30)	(2340, 2)	(2340, 2)	(468, 6)	(468, 6)	(4680, 26)	(180, 4)
4255 = 5.37.23	3, 1	(720, 6)	(180, 2)	(180, 2)	(36, 2)	(36, 2)	(144, 6)	(72, 12)
4355 = 13.5.67	2, 1	(3960, 6)	(180, 6)	(900, 18)	(660, 2)	(660, 2)	(120, 2)	(60, 4)
4395 = 5.293.3	3, 1	(7920, 2)	(220, 2)	(220, 2)	(180, 2)	(180, 2)	(40, 4)	(80, 2)
4755 = 317.5.3	2, 1	(1560, 6)	(60, 6)	(60, 6)	(156, 6)	(156, 6)	(12, 12)	(24, 6)
5395 = 5.13.83	2, 1	(2040, 6)	(204, 2)	(204, 2)	(60, 2)	(60, 2)	(24, 6)	(12, 12)
5495 = 157.5.7	3, 1	(336, 6)	(84, 6)	(84, 6)	(84, 6)	(84, 6)	(336, 6)	(168, 12)
6055 = 173.5.7	3, 1	(1008, 42)	(252, 2)	(252, 2)	(252, 6)	(252, 6)	(144, 2)	(72, 4)

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REFERENCES

- [1] A. Azizi, *Unités de certains corps de nombres imaginaires et abéliens sur \mathbb{Q}* , Ann. Sci. Math. Québec **23** (1999), no 1, 15-21.
- [2] A. Azizi, *Sur la capitulation des 2-classes d'idéaux de $k = \mathbb{Q}(\sqrt{2pq}, i)$, où $p \equiv -q \equiv 1 \pmod{4}$* Acta Arith. **94** (2000), 383-399.
- [3] A. Azizi, *Sur une question de capitulation*, Proceedings of the American Mathematical Society, volume **130** (2002), 2197-2202.
- [4] A. Azizi, *Sur les unités de certains corps de nombres de degré 8 sur \mathbb{Q}* , Ann. Sci. Math. Québec **29** (2005), no. 2, 111-129.
- [5] A. Azizi et M. Taous, *Détermination des corps $k = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ dont les 2-groupes de classes sont de type $(2, 4)$ ou $(2, 2, 2)$* , Rend. Istit. Mat. Univ. Trieste. **40** (2009), 93-116.
- [6] A. Azizi, A. Zekhnini et M. Taous, *On the unramified quadratic and biquadratic extensions of the field $\mathbb{Q}(\sqrt{d}, i)$* , IJA, Volume **6**, No. 24 (2012), 1169-1173.
- [7] A. Azizi, A. Zekhnini et M. Taous, *On the generators of the 2-class group of the field $k = \mathbb{Q}(\sqrt{d}, i)$* , IJPAM, Volume **81**, No. 5 (2012), 773-784.
- [8] A. Azizi, A. Zekhnini et M. Taous, *On the 2-class field tower of $\mathbb{Q}(\sqrt{2p_1p_2}, i)$ and the Galois group of its second Hilbert 2-class field.*, Collectanea Mathematica **65** (2014), no. 1, 131-141.
- [9] E. Benjamin, F. Lemmermeyer, C. Snyder, *Real Quadratic Fields with Abelian 2-Class Field tower*, Journal of Number Theory, Volume **73**, Number 2, December (1998), pp. 182-194 (13).
- [10] E. Benjamin, F. Lemmermeyer, C. Snyder, *Imaginary quadratic fields with $Cl_2(k) \simeq (2; 2; 2)$* , J. Number Theory **103** (2003), 38-70.
- [11] H. U. Besche, B. Eick, and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP 4 package, available also in MAGMA.
- [12] G. Gras, *Class field theory, from theory to practice*, Springer Verlag (2003).
- [13] D. Hilbert, *Ueber den Dirichlet'schen biquadratischen Zahlkörper*, Math. Annalen **45** (1894), 309-340.
- [14] P. Kaplan, *Sur le 2-groupe de classes d'idéaux des corps quadratiques*, J. Reine angew. Math. **283/284** (1976), 313-363.
- [15] F. Lemmermeyer, *On 2-class field towers of imaginary quadratic number fields*, Journal de Théorie des Nombres de Bordeaux **6** (1994), 261-272.
- [16] S. Lang, *Algebraic Number Theory*, Second Edition, Springer, New York, 1994.
- [17] F. Lemmermeyer, *Ideal class groups of cyclotomic number fields I*, Acta Arithmetica **72** (1995), no. 4, 347-359.
- [18] F. Lemmermeyer, *On 2-class field towers of some imaginary quadratic number fields*, Abh. Math. Sem. Hamburg **67** (1997), 205-214.

- [19] F. Lemmermeyer, *Reciprocity Laws*, Springer Monographs in Mathematics, Springer-Verlag, Berlin 2000.
- [20] D. C. Mayer, *The distribution of second p -class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456.
- [21] T. M. McCall, C. J. Parry, R. R. Ranalli, *On imaginary bicyclic biquadratic fields with cyclic 2-class group*, J. Number Theory **53**, (1995), 88-99.
- [22] M. F. Newman and E. A. O'Brien, *Classifying 2-groups by coclass*, Trans. Amer. Math. Soc. **351** (1999), 131–169.
- [23] H. Nover, *Computation of Galois groups associated to the 2-class towers of some imaginary quadratic fields with 2-class group $C_2 \times C_2 \times C_2$* , J. Number Theory **129** (2009), 231–245.
- [24] The PARI Group, PARI/GP, Bordeaux, Version 2.4.4 (beta), May 9 2011, (<http://pari.math.u-bordeaux.fr>).
- [25] A. Scholz, *Über die Lösbarkeit der Gleichung $t^2 - Du^2 = -4$* , Math. Z. **39** (1934), 95-111.
- [26] O. Taussky, *A remark concerning Hilbert's Theorem 94*, J. Reine Angew. Math. **239/240** (1970), 435-438.
- [27] H. Wada, *On the class number and the unit group of certain algebraic number fields*, J. Fac. Univ. Tokyo Sect. I **13** (1966), 201-209.

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